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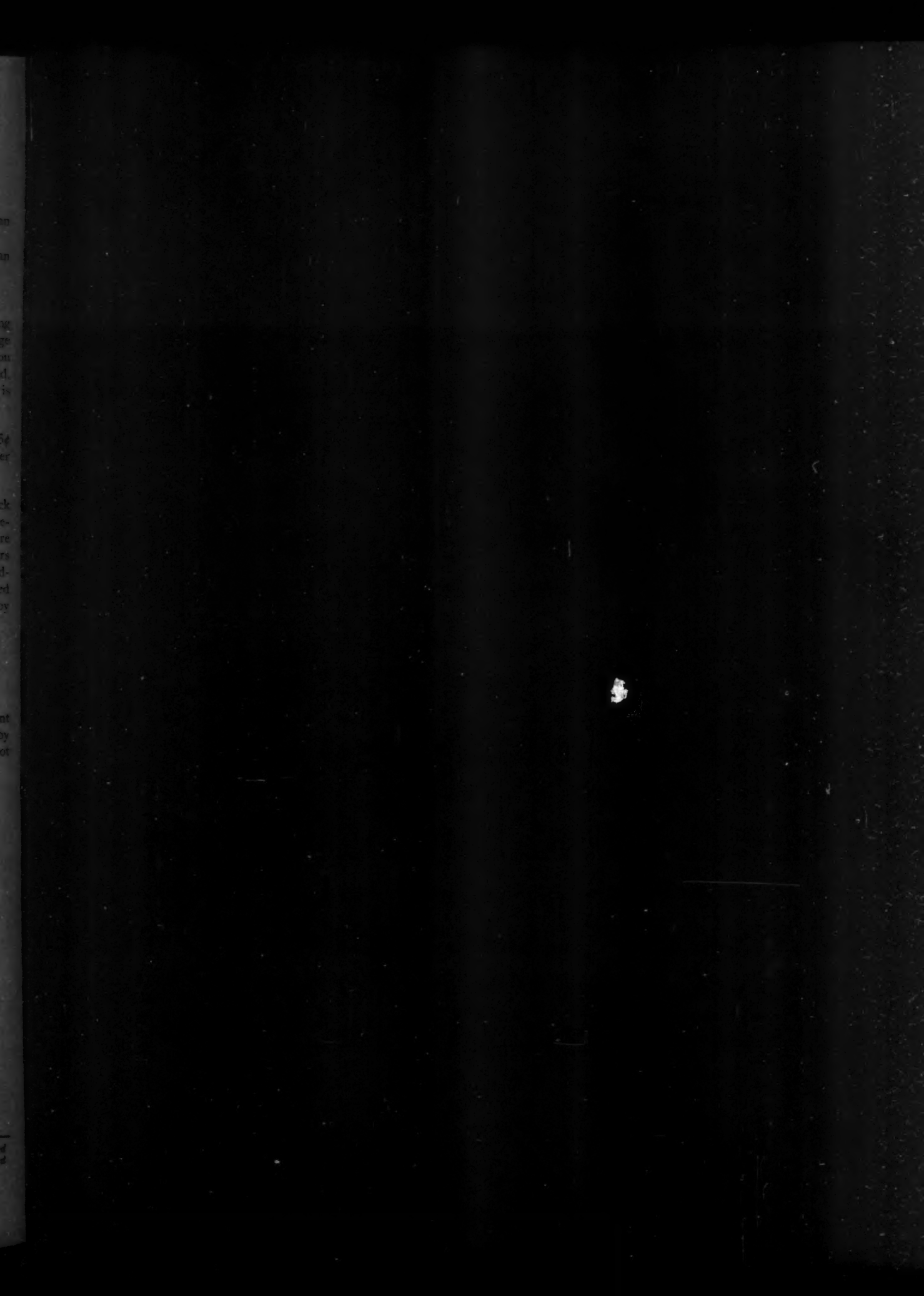

Mathematical Reviews

Vol. 7, No. 3 — MARCH, 1946 — Pages 105-144

HISTORY

***Devarāja. Kuṭṭākāraśiromaṇi.** Ānandāśrama Sanskrit Series no. 125. Baḷavant Dattātreya Āpṭe, Poona, 1944. 53 pp. (Sanskrit)

The Kuṭṭākāraśiromaṇi is a treatise on the Hindu method of obtaining a solution in positive integers of an indeterminate equation of the first degree. The work was written by Devarāja, a commentator of the author Aryabhaṭa, and contains Devarāja's own commentary. This is the treatise referred to by Datta and Singh [History of Hindu Mathematics, part II, Lahore, 1938, p. 88] as existing in four manuscript copies in the Oriental Library at Mysore. The subject of first degree indeterminate analysis is frequently called kuṭṭākāra by the Hindus. Apparently it was first treated in the Āryabhaṭīya by Āryabhaṭa, born 476 A.D. An exposition is found in the book by Datta and Singh. The text itself states problems leading to first degree indeterminate equations. The commentary following the problem gives the method of solution. Reference is made to previous Hindu treatises such as the Āryabhaṭīya and the Līlāvatī of Bhāskara. This text should be of great interest to students of the history of mathematics. An introduction in a European language would be very helpful to scholars in the western world. *E. B. Allen* (Troy, N. Y.).

***Muñjāla. Laghumānasaṃ.** Ānandāśrama Sanskrit Series no. 123. Baḷavant Dattātreya Āpṭe, Poona, 1944. 32 pp. (Sanskrit)

This is the text of an astronomical work written by Muñjāla in the Śaka year 854, approximately 932 A.D. A brief discussion of the text and its author can be found in G. Thibaut, Astronomie, Astrologie und Mathematik, Grundriss der Indo-Arischen Philologie und Altertumskunde III 9, 1899, p. 59. The work is written in the usual Hindu style, namely, a śloka verse, commentary on the verse, and so on. The commentary is by Parameśvara. The text is divided into sections with the following titles: the mean motions, the true motions, miscellaneous rules, the three inquiries (that is, direction, place and time), eclipses, and the operation for apparent longitude. Most of these titles occur also as chapter headings in the Sūrya Siddhānta. An introduction in some European language, containing a discussion of the author and commentator, the manuscripts used and the contents of the book, would be helpful to those who are interested in the history of science but cannot read Sanskrit. *E. B. Allen* (Troy, N. Y.).

Krishnaswami Ayyangar, A. A. Peeps into India's mathematical past. J. Mysore Univ. Sect. A. 5, 101–115 (1945). [MF 13913]

Neugebauer, O. Studies in ancient astronomy. VII. Magnitudes of lunar eclipses in Babylonian mathematical astronomy. Isis 36, 10–15 (1945). [MF 14043]

Continuation of part III of this series [Quellen und Studien zur Geschichte der Mathematik, Abt. B, Studien 4, 193–346 (1938)].

Sarton, George. Remarks on the study of Babylonian mathematics. Isis 31, 398–404 (1940). [MF 13981]

Gandz, Solomon. Studies in Babylonian mathematics. II. Conflicting interpretations of Babylonian mathematics. Isis 31, 405–425 (1940). [MF 13982]

This paper began as a review of Thureau-Dangin, Textes Mathématiques Babyloniens [Leiden, 1938]. The "conflicting interpretations" are in part improvements of interpretations obtained by Thureau-Dangin beyond the reviewer's first edition of a few difficult passages; in part the conflicts are merely apparent, being caused by differences in terminology. *O. Neugebauer* (Princeton, N. J.).

von Fritz, Kurt. The discovery of incommensurability by Hippasus of Metapontum. Ann. of Math. (2) 46, 242–264 (1945). [MF 12400]

The discovery of the irrational numbers is one of the most essential problems in the history of Greek mathematics. Since Vogt's first critical evaluation [Bibliotheca Math. (3) 10, 97–155 (1910)] historians have derived widely different pictures from the incomplete and contradictory ancient tradition for the history of the irrationals. It might be said, however, that the trend in later years has been directed toward condensing the whole development into a rather short time interval around 400 B.C., mainly following a book by E. Frank [Platon und die Sogenannten Pythagoreer, Niemeyer, Halle, 1923]. One of the essential points in Frank's book was the sharp distinction between the philosophical sect of the Pythagoreans and the "so-called" Pythagoreans of a much later period whose main figure appears to have been Archytas of Tarentum. The second group alone is supposed to have played a role in the development of mathematics. The present article attempts to invalidate these conclusions, at least in part. The term "so-called" is interpreted as emphasizing only the unusual naming of the group after a person and not after a locality. The link between the older group of Pythagoreans and the later mathematicians is seen in Hippasus of Metapontum, who is credited with the discovery of irrational magnitudes. Frank [appendix 19 in the book cited] underlined very strongly the weakness of the ancient reports concerning Hippasus and the reviewer feels that Frank's arguments cannot be dismissed so easily. The author goes even further and proposes a restoration of the main steps in Hippasus' conclusions, assigning a major role to the investigation of the regular pentagon. The present article has interesting points of contact with an article of van der Waerden [Math. Ann. 117, 141–161 (1940); these Rev. 1, 289] which was apparently not accessible to the author.

O. Neugebauer (Princeton, N. J.).

Tannery, Paul, et Zeuthen, H. G. Trois lettres inédites de la correspondance Paul Tannery-H. G. Zeuthen. Revue Sci. (Rev. Rose Illus.) 80, 99–103 (1942). [MF 13816]

These letters deal with questions in the Greek theory of

conic sections and the significance of certain constructions by means of "neusis." *O. Neugebauer.*

Brun, Viggo. Quadrature of the circle. Norsk Mat. Tidsskr. **23**, 13 pp. (1941). (Norwegian) [MF 13361]
A historical lecture. A dozen assorted formulas involving π are quoted at the end.
The paper is identical with that quoted in this Rev 3, 97.

García de Zuñiga, E. Galileo and pure mathematics. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 171–174 (1945). (Spanish) [MF 13728]

Boyer, Carl B. Fermat's integration of X^n. Nat. Math. Mag. **20**, 29–32 (1945). [MF 13979]

Loria, Gino. Perfectionnements, évolution, métamorphoses du concept de "coordonnées." Contribution à l'histoire de la géométrie analytique. Mathematica, Timişoara **21**, 66–83 (1945). [MF 13973]
Completion of an article of which the first two parts appeared in the same journal **18**, 125–145 (1942); **20**, 1–22 (1944); these Rev. **4**, 65; **6**, 141.

Dugas, René. Le principe de la moindre action dans l'oeuvre de Maupertuis. Revue Sci. (Rev. Rose Illus.) **80**, 51–59 (1942). [MF 13823]

Dugas, René. Sur l'origine du théorème de Coriolis. Rev. Sci. (Rev. Rose Illus.) **79**, 267–270 (1941). [MF 13321]

220th Anniversary of the Academy of Sciences of the USSR and the development of the science of mechanics. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 185–192 (1945). (Russian and English) [MF 14030]

***Ioffe, A. F., Editor. Očerki po Istorii Akademii Nauk. Fiziko-Matematičeskie Nauki. [Outline of the History of the Academy of Sciences. Physico-Mathematical Sciences.]** Akademiya Nauk SSSR, Moscow-Leningrad, 1945. 78 pp. (Russian)
The book contains separate accounts of physics by S. I. Vavilov, mathematics by the editor, and astronomy by A. A. Mihaĭlov.

***Kuznecov, B. G. Lomonosov, Lobačevskiĭ, Mendeleev. Očerki Žizni i Mirovozzreniya. [Lomonosov, Lobačevskiĭ, Mendeleev. Outline of Life and World Outlook].** Akademiya Nauk SSSR, Moscow-Leningrad, 1945. 334 pp. (Russian)

Lundmark, Knut. Nicolaus Kopernikus (Kopernik) and his astronomical reformation. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 3, 22–39 (1945). [MF 13542]

Hadamard, Jacques. Obituary: George David Birkhoff. C. R. Acad. Sci. Paris **220**, 719–721 (1945). [MF 14064]

Sarton, George. Grassmann—1844. Isis **35**, 326–330 (1 plate) (1945). [MF 12968]

Machado, Bernardino. David Hilbert. Gaz. Mat., Lisboa **4**, no. 14, 1–2 (1943). (Portuguese) [MF 12964]

Carleman, Torsten. Annonce de la mort de Erik Albert Holmgren. Acta Math. **76**, i–iii (1945). [MF 13196]
Contains a list of 32 papers by Holmgren.

Obituary: Nikolaĭ Evgrafovič Kočin [Nikolai Kochin]. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 3–12 (1945). (Russian and English) [MF 13503]
Contains a list of Kočin's publications.

Fayet, J. Obituary: Henri Lebesgue. 1875–1941. Revista Mat. Hisp.-Amer. (4) **1**, 195–197 (1941). (Spanish) [MF 12997]

Vicente Gonçalves, J. Henri Lebesgue. Gaz. Mat., Lisboa **3**, no. 12, 2–3 (1942). (Portuguese) [MF 12960]

de Mira Fernandes, A. Sophus Lie. Gaz. Mat., Lisboa **3**, no. 12, 1–2 (1942). (Portuguese) [MF 12959]

Cartwright, M. L. Obituary: Grace Chisholm Young. J. London Math. Soc. **19**, 185–192 (1944). [MF 13641]
Contains a list of G. C. Young's papers.

ALGEBRA

Chakrabarti, S. C. Some further algebraic identities. J. Indian Math. Soc. (N.S.) **8**, 115–119 (1944). [MF 13275]
The author considers a number of identities involving the well-known generalization of the binomial coefficients

$$ {}_kS_n(a) = {}_kS_n = \frac{(a^k-1)(a^{k-1}-1)\cdots(a^{k-n+1}-1)}{(a^n-1)(a^{n-1}-1)\cdots(a-1)} a^{n(n-1)/2}. $$

We have $\lim_{a\to 1} {}_kS_n(a) = \binom{k}{n}$, and ${}_kS_n$ satisfies the fundamental difference equation ${}_{k+1}S_n = a^n {}_kS_n + a^{n-1} {}_kS_{n-1}$. When k is an integer, ${}_kS_n(a)$ is the nth elementary symmetric function of $1, a, a^2, \cdots, a^k$. A typical result is

$$ \sum_{m=0}^{r} {}_rS_{r-m}\ {}_pS_m = {}_{r+p}S_r. $$

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Riordan, John. Permutations without 3-sequences. Bull. Amer. Math. Soc. **51**, 745–748 (1945). [MF 13614]
The author enumerates permutations of $1, \cdots, n$ which contain none of the sequences $123, 234, \cdots, n-2\ n-1\ n$, or more generally which contain a specified number of them. A table for $n \leqq 10$ is appended. *I. Kaplansky.*

Finan, E. J. Magic rectangles modulo p. Amer. Math. Monthly **52**, 502–506 (1945). [MF 14086]
Write an odd prime p as $1+kt$, where k is even and $(k, t)=1$. Let α and β belong to the exponents k and t (mod p). The author constructs a k by t rectangle from elements $\alpha^i\beta^j$ (mod p). The title is inspired by the fact that the sum of the elements in any row or column is 0 (mod p). Various other properties of the array are proved.
I. Kaplansky (Chicago, Ill.).

Hall, Marshall. An existence theorem for Latin squares. Bull. Amer. Math. Soc. **51**, 387–388 (1945). [MF 12516]
Given a Latin rectangle, that is, an arrangement of the numbers $1, 2, \cdots, m$ into k rows of m numbers each in such a way that no number occurs twice in any row or column, then it is always possible to add $m-k$ rows in such a way that the resulting square is a Latin square. The proof uses

a theorem of P. Hall [J. London Math. Soc. 10, 26–30 (1935)]. *H. B. Mann* (Columbus, Ohio).

Finney, D. J. Some orthogonal properties of the 4×4 and 6×6 Latin squares. Ann. Eugenics 12, 213–219 (1945). [MF 13925]

This paper deals with the problem of partitioning the n^2 cells of an $n\times n$ Latin square into sets of $nk_1, nk_2, \cdots$, where $k_1+k_2+\cdots=n$, in such a way that the ith set has k_i cells in each row, k_i cells in each column, and k_i cells of each letter. A solution is described as a $(k_1, k_2, \cdots)$ orthogonal partition of the Latin square. The classical instance is a $(1, 1, \cdots)$ or (1^n) partition, which is a Graeco-Latin or Eulerian square. The arrangement

A	b	c	D
B	a	d	C
c	D	A	b
d	C	B	a

is a (2, 2) or (2^2) partition of a 4×4 square, because each row or column contains two small and two capital letters, while each of the four letters occurs twice in either style. Although no (1^6) partition exists for any 6×6 Latin square, there exist 6×6 Latin squares possessing every other kind of orthogonal partition. Moreover, every one of the 6!5!9408 Latin squares of that order admits at least one (2^3) partition. *H. S. M. Coxeter* (Toronto, Ont.).

Fisher, R. A. A system of confounding for factors with more than two alternatives, giving completely orthogonal cubes and higher powers. Ann. Eugenics 12, 283–290 (1945). [MF 13926]

Let $\alpha, \beta, \cdots$ denote the n generators of the Abelian group of order p^n and type $(1, \cdots, 1)$. Let Latin letters $A, B, \cdots$ be associated with $q=(p^n-1)/(p-1)$ elements, one chosen arbitrarily from each subgroup of order p; let each generator be represented as a formal product $A^aB^b\cdots$, where $a, b, \cdots$ (with possible values $0, 1, \cdots, p-1$) are the powers to which that generator appears in $A, B, \cdots$, respectively. If the Latin letters $A, B, \cdots$ are regarded as generators of an Abelian group of order p^q and type $(1, \cdots, 1)$, then every element of the original group (save the identity) will appear as a product of powers of p^{n-1} Latin letters, thus exhibiting the group of order p^n as a subgroup in the group of order p^q. For instance, if the 8 letters A, B, C, D, E, F, G, H are associated with the respective elements α, β, $\alpha\beta$, $\alpha\beta^2$, $\alpha\beta^3$, $\alpha\beta^4$, $\alpha\beta^5$, $\alpha\beta^6$ of the Abelian group of order 49, we write $\alpha=ACDEFGH$, $\beta=BCD^2E^3F^4G^5H^6$, and deduce $\alpha\beta=ABC^3D^4E^5F^6G^6$, $\alpha\beta^2=AB^2C^4D^6E^8F^8G^4H^6$, etc.

The consequent schemes of Latin letters are used to construct symmetrical block designs of various kinds. The case when $p=2$ was discussed in an earlier paper [same Ann. 11, 341–353 (1942); these Rev. 4, 127]. The paper closes with a further generalization, using subgroups of order p^r instead of p. *H. S. M. Coxeter* (Toronto, Ont.).

Raymond, François. Sur certains déterminants intervenant dans la théorie des oscillations de systèmes symétriques. Revue Sci. (Rev. Rose Illus.) 80, 128–129 (1942). [MF 13817]

The author proves the well-known theorem on the value of a circulant and applies it to a few special cases.

J. Williamson (Flushing, N. Y.).

Blumenthal, L. M. Note on an extension of matrix rank. Univ. Nac. Tucumán. Revista A. 4, 235–241 (1944). [MF 13026]

A matrix $\mathfrak{R}=(r_{ij})$ is defined to have a relative rank ρ with respect to a class $\{\mathfrak{E}\}$ of matrices if there exists a matrix $\mathfrak{E}=(\epsilon_{ij})$ of the class such that the matrix $(\epsilon_{ij}r_{ij})$ has rank ρ. If the class consists of the single matrix each element of which is 1, the relative rank of $\mathfrak{R}$ with respect to that class is the ordinary rank of $\mathfrak{R}$. The author considers the relative rank of a real symmetric matrix $\mathfrak{R}$ of order n, where $r_{ii}=1$, with respect to the class $\{\mathfrak{E}\}$ of all symmetric matrices $\mathfrak{E}$ of order at most n with $\epsilon_{ii}=1$ and $\epsilon_{ij}=\pm1$. He proves several theorems of which the following is typical. Let $\mathfrak{R}=(r_{ij})$, $r_{ij}=r_{ji}$, $r_{ii}=1$, $0<r_{ij}<1$ $(i\neq j)$, $i, j=1, \cdots, n$; $n>4$. If each third-order principal minor of $\mathfrak{R}$ has a relative rank 2 with respect to the class $\{\mathfrak{E}\}$, then $\mathfrak{R}$ has also unless each element of $\mathfrak{R}$ has either the value $\frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$. The proofs are geometric in character and are based on the author's previous work. *J. Williamson* (Flushing, N. Y.).

Ghosh, N. N. On a new reduction theorem of matrices. Bull. Calcutta Math. Soc. 37, 33–36 (1945). [MF 13779]

Let M be an m by n matrix of rank r. The author shows how M can be expressed in the form (1) $M=\sum_{h=1}^{r}C^hR_h$, where C^h is a column vector and R_h a row vector belonging, respectively, to the column and row spaces of M. He defines the auxiliary matrix of class s based on a square submatrix N, of order s, of M to be the matrix whose element in the ith row and jth column is the product of $|N|^{-1}$ by the determinant of the matrix of order $s+1$ obtained by bordering N by the ith row and the jth column of M. By means of determinantal identities he expresses the difference between an auxiliary matrix of class s and ones of lower class in terms of matrices C^hR_h and by successive applications obtains (1). *J. Williamson.*

Parker, W. V. The characteristic roots of matrices. Duke Math. J. 12, 519–526 (1945). [MF 13521]

If A is an $n\times n$ matrix with complex elements, and $\bar{A}'$ is its conjugate transpose, then $A\bar{A}'$ has real nonnegative characteristic roots $\rho_1^2, \cdots, \rho_n^2$ $(\rho_i\geqq0)$. The author shows that, for any $k=1, \cdots, n$, any two integers p and s such that $ps\equiv0 \pmod n$, and any arrangement of the ρ's, there exists a unitary matrix ψ such that $A\psi$ has the characteristic roots $(\prod_q\rho_{q+k})^{1/p}$, $q=1, \cdots, p-1$. In particular, there is a unitary matrix ψ such that $(A\psi)^n=|A|\cdot I$. If A is of rank $r<n$, there is a unitary matrix ψ such that $(A\psi)^p=0$ for $p\geqq n/(n-r)$. The principal theorem states that, if x and y are variable vectors such that $x\bar{x}'=y\bar{y}'=1$, then $xA\bar{y}'$ is the set of all complex numbers whose absolute values do not exceed the largest of the numbers $\rho_1, \cdots, \rho_n$.

C. C. MacDuffee (Madison, Wis.).

Barankin, Edward W. Bounds for the characteristic roots of a matrix. Bull. Amer. Math. Soc. 51, 767–770 (1945). [MF 13617]

Let $A=(a_{rs})$ be a square matrix of order n over the complex field and let λ be a characteristic root of A. Let

$$R_r=\sum_s|a_{rs}|, \quad T_s=\sum_r|a_{rs}|, \quad R_r^{(k)}=\sum_s|a_{rs}|^k$$

and

$$T_s^{(k)}=\sum_r|a_{rs}|^k.$$

The author proves the following theorems: (1) $|\lambda|^2$ is bounded above by $\max_r(R_rT_r)$; (2) for each fixed k, $0\leqq k\leqq2$, $|\lambda|^2$ is bounded above by $\max_r(R_r^{(k)}T_r^{(2-k)})$; (3) $|\lambda|$ is bounded above both by $\max_r R_r$ and by $\max_s T_s$.

Theorem (1) establishes a bound conjectured by A. B. Farnell and proved by him for the case that $n=2$ [same Bull. 50, 789–794 (1944); these Rev. 6, 113]. *J. Williamson.*

Farnell, A. B. Limits for the field of values of a matrix. Amer. Math. Monthly 52, 488–493 (1945). [MF 14083]

Let $A=(a_{rs})$ be a square matrix of order n over the complex field. If x is a column vector of unit norm ($\bar{x}'x=1$), the numbers $\bar{x}'Ax$ form the field of values of A. In a previous paper the author proved the following two theorems [Bull. Amer. Math. Soc. 50, 789–794 (1944); these Rev. 6, 113]. If λ is in the field of values of A, (1) $\lambda \leqq (\sum_{r,s}|a_{rs}|^2)^{\frac{1}{2}}$; (2) $|\lambda| \leqq (\sum_{r,s} d_{rs}^2)^{\frac{1}{2}}$, $d_{rs}=\frac{1}{2}(|a_{rs}|+|a_{sr}|)$. From these he deduces: (3) if (1) is applied to the sequence of matrices A^m, $m=2^q$, $q=0, 1, \cdots$, a monotone sequence of numbers is obtained which converges to the absolute value of the dominant characteristic root δ of A; (4) if (2) is applied to the same sequence of matrices, the sequence of numbers obtained converges to $|\delta|$. Since there exists a unitary matrix P such that $PAP'=B$, where B is a triangle matrix, and since $\sum_{r,s}|a_{rs}|^2=\sum_{r,s}|b_{rs}|^2$, it is only necessary to prove the theorems for triangle matrices. The author does this but because of the complexity of notation gives the details only when $n=2$. By using the companion matrix of a polynomial these theorems may be applied to the dominant root of any polynomial equation. *J. Williamson.*

Lagerstrom, Paco. A proof of a theorem on commutative matrices. Bull. Amer. Math. Soc. 51, 535–536 (1945). [MF 12813]

A proof in the notation of endomorphisms of the known theorem that a matrix which commutes with every matrix commutative with the matrix A is a scalar polynomial in A. A possible generalization is noted. *C. C. MacDuffee.*

Papy, Georges. Sur un lemme de M. J. Radon. Bull. Soc. Roy. Sci. Liége 11, 479–487 (1942). [MF 13117]

The author proves the following theorem. If A and B are two m by n matrices over a field of characteristic different from two and if the rank r of (A, B) satisfies $r \leqq n$, then there exists a symmetric orthogonal matrix S all of whose elements are 1, -1 or 0 such that $A-BS$ and $AS-B$ both have rank r. This is a generalization of a theorem of Radon: if A and B are square matrices over the complex field there exists a symmetric matrix S such that $A-BS$ has the same rank as (A, B) [Monatsh. Math. Phys. 48, 198–204 (1939); these Rev. 1, 98]. The proof is purely algebraic in character. *J. Williamson* (Flushing, N. Y.).

Papy, Georges. Formes biquadratiques et matrices. Bull. Soc. Roy. Sci. Liége 12, 88–97 (1943). [MF 13132]

Consider the biquadratic form $\mathfrak{B}=l_{ijmn}\xi_i\xi_j\eta_m\eta_n$, where the coefficients l_{ijmn} and the variables $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n$ range over an arbitrary field. Unless one of the integers m, μ is two and the other odd, $\mathfrak{B}$ can be written as a linear combination of the squares of $m\mu$ linearly independent bilinear forms in the ξ and η. If $m=2$ and μ is odd, the theorem is true provided one of the quadratic forms $l_{imn}\xi_i\xi_j$ is nonsingular. *C. C. MacDuffee* (Madison, Wis.).

Egan, M. F. Symmetric matrices and quadratic forms. Math. Gaz. 29, 89–91 (1945). [MF 13334]

In the process of reducing a quadratic form to a sum of squares by means of triangular substitutions, the author obtains seven known theorems concerning symmetric matrices. *C. C. MacDuffee* (Madison, Wis.).

Browne, E. T. Concerning a certain ring of homographies. J. Elisha Mitchell Sci. Soc. 61, 19–32 (1945). [MF 12984]

This is a continuation of an earlier paper by the author [Ann. of Math. (2) 29, 483–492 (1928)]. There exist symmetrical triples A, B, C of involutory matrices of order $n=2m$ such that for any matrix $H=xA+yB+zC$, where x, y, z are complex,

$$H^2=k^2I, \quad k^2=f(x, y, z)=x^2+y^2+z^2-2xy-2xz-2yz.$$

The matrices A, B, C, I form a basis for the total matric algebra of order 2^2 (complex quaternions) whose constants of multiplication are rational integers. Two involutions H and $H'=x'A+y'B+z'C$ are harmonic if the points (x, y, z) and (x', y', z') are conjugate with respect to the conic $f(x, y, z)=0$. There are exactly two symmetrical triples which are harmonic to A, B, C, respectively. If Σ is a nonsingular quadric and if G is a line reflection whose axes are conjugate lines as to Σ, there exist two symmetrical triples of involutions A, B, C and A', B', C' ($n=2$), the first triple having as axes lines of one regulus on Σ, the second having as axes lines of the conjugate regulus, such that G is expressible in the form $G=(xA+yB+xC)(x'A'+y'B'+z'C')$. *C. C. MacDuffee* (Madison, Wis.).

Bouianger, J. Sur l'équation de Leuschner. Bull. Soc. Roy. Sci. Liége 12, 680–692 (1943). [MF 13158]

The equation is

$$(x-m)^2(x^2-2x\cos\Psi+1)^2-m^2=0.$$

The location of its roots is discussed for various values of the parameters.

Segre, B. The algebraic equations of degrees 5, 9, 157, $\cdots$, and the arithmetic upon an algebraic variety. Ann. of Math. (2) 46, 287–301 (1945). [MF 12403]

The author is concerned with theorems of the following type. Theorem H_i: the solution of the general equation $\sum_0^n a_k x^{n-k}=0$ for $n>n_i$ is derivable from algebraic functions of a single variable and a number, independent of n, of algebraic functions of at most $n-i-3$ variables. Theorems H_1 and H_2 have been proved by Hilbert, with $n_1=5$ and $n_2=9$, and the author takes the next step with $n_3=157$, using only one function of $n-6$ variables. He also improves on Hilbert's H_2, which uses one function of four variables and one of $n-5$, by eliminating the necessity of the former function.

Theorem H_i follows from theorem B_i: for $n>n_i$, if a Tschirnhausen transformation $y=\sum_1^n t_k x^k$ is effected on the equation $\sum_0^n a_k x^{n-k}=0$ to give $\sum_0^n b_k(t)y^{n-k}=0$, then a set of t_k's can be chosen so that $b_1=\cdots=b_{i+2}=0$, the t_k's being in a field obtainable from $R(a_0, \cdots, a_n)$ by adjunction of a finite number of algebraic functions of a single argument. Theorems B_1, B_2 and B_3 with $n_1=5$, $n_2=9$, and $n_3=157$, are proved by considering the intersection, in the space of coordinates t_k, of the algebraic varieties defined by $b_k(t)=0$. It is shown that some point in the intersection of $b_1(t)=0$, $\cdots$, $b_{i+2}(t)=0$ can be obtained by suitable adjunctions to the field of coefficients. *R. J. Walker.*

Brauer, Richard. A note on systems of homogeneous algebraic equations. Bull. Amer. Math. Soc. 51, 749–755 (1945). [MF 13615]

The author considers systems of equations

$$f_i(x_1, \cdots, x_n)=0, \qquad i=1, \cdots, h, \tag{1}$$

where each f_i is a homogeneous polynomial of degree r_i with

coefficients in a field K. He proves that, if m is any integer and the number n of unknowns exceeds a certain bound $\Phi(r_1, \cdots, r_h, m)$, then we can find a soluble extension field K_2 of K in which (1) has an m-dimensional linear family of solutions. The degree $(K_2:K)$ is less than a bound depending only on the r_i, and each prime factor of $(K_2:K)$ is at most equal to $\max(r_i)$. The proof is by induction: (1) is reduced to an equation

$$a_1u_1^s+a_2u_2^s+\cdots+a_tu_t^s=0, \tag{2}$$

together with a system of lower degree. If K has the property that, for some function $\Psi(s)$, every equation (2) with $t \geqq \Psi(s)$ has a nontrivial solution, then K itself contains an m-dimensional linear family of solutions whenever n exceeds a bound of the above type. All $\mathfrak{P}$-adic number fields have this property.

These theorems have application to Hilbert's resolvent problem: if l_n is the least number such that the roots of the general equation of degree n can be expressed in terms of the coefficients by using algebraic functions of l_n parameters, then it follows that $\lim_{n\to\infty}(n-l_n)=\infty$. *G. Whaples.*

Vessiot, Ernest. Sur une théorie nouvelle de la réductibilité des équations algébriques. Ann. Sci. École Norm. Sup. (3) **58**, 1–36 (1941). [MF 12897]

The author has announced his results previously [C. R. Acad. Sci. Paris **210**, 159–161 (1940); these Rev. **1**, 194]. The present paper contains a detailed account.

R. Brauer (Toronto, Ont.).

Tchernikow, S. A generalization of Kronecker-Capelli's theorem on systems of linear equations. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 437–448 (1944). (Russian. English summary) [MF 12306]

This paper deals with the existence of solutions of systems of inequalities which may be considered as generalizations of systems of linear equations: whereas in the latter case we are interested in the values of the unknowns which will make given linear forms take given values, here the unknowns must make the linear forms differ from given values by not more than preassigned amounts ϵ_i (when all ϵ_i are zero we have the special case of linear equations). The author arrives at necessary and sufficient conditions for consistency of inequalities which in the special case of equations may be reduced to the theorem mentioned in the title, according to which consistency is equivalent to equality of ranks of the matrix of the coefficients and the augmented matrix. To achieve this result the author works with what he calls reduced systems and with values of the unknowns which when substituted into the linear forms make them differ from given values exactly by the amounts ϵ_i. Coefficients and unknowns are complex numbers.

G. Y. Rainich (Ann Arbor, Mich.).

Doyle, T. C. Euclidean metric invariants of conics by tensor algebra. Amer. Math. Monthly **52**, 179–187 (1945). [MF 12254]

This is a short description of the connection between the notation and technique of tensor algebra on the one hand and of the symbolic methods in classical invariant theory on the other. The results given by Weitzenböck [Akad. Wiss. Wien, S.-B. II. **122**, 1241–1258 (1913)] for the invariant system of a conic, for rotations and translations, are here exhibited in tensor form. The two notations for each concomitant are compared: for example,

$$(xa)(a\,|\,b)(bx)=g_{QR}\delta^{rs}g_{sR}x^Qx^R.$$

[Cf. D. E. Littlewood, Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41 for a general treatment of invariants by tensor algebra.]

H. W. Turnbull (St. Andrews).

Abstract Algebra

Eilenberg, Samuel, and MacLane, Saunders. General theory of natural equivalences. Trans. Amer. Math. Soc. **58**, 231–294 (1945). [MF 13310]

There are in group theory a number of standard methods for constructing new groups $T(A, B, \cdots)$ from given groups $A, B, \cdots$. We have, for example, $A\times B$ (direct product); Hom (A, B) (the group of homomorphisms of A into B, A and B Abelian); Ext (A, B) (the group of extensions of A by B, A and B Abelian); $C(A)$ (the commutator subgroup of A). It frequently happens that the construction $T(A, B, \cdots)$ admits a law of transformation; that is, every system of homomorphisms $A_1\to A_2$, $B_1\to B_2$, $\cdots$ induces a unique homomorphism (1) $T(A_1, B_1, \cdots)\to T(A_2, B_2, \cdots)$. Then T is called a functor and is said to be covariant in the arguments $A, B, \cdots$; T might equally well be contravariant in one or more arguments, for example in A: the law of transformation then reads $T(A_2, B_1, \cdots)\to T(A_1, B_2, \cdots)$. The examples above are all functors, but $Z(A)$ (the center of A) is not, unless the allowable homomorphisms $A_1\to A_2$ are suitably restricted. The arguments and values of a functor need not be groups but can be taken from abstract classes of objects plus mappings, which the authors call categories. Thus the theory of functors is also applicable to abstract sets, linear spaces, and so on. In the case of groups, mappings are understood to be homomorphisms.

Let $S(A, B, \cdots)$ be a functor which has the same variance as the functor $T(A, B, \cdots)$ in each argument. Suppose there is a one-to-one mapping λ, a function of $A, B, \cdots$, of $T(A, B, \cdots)$ onto $S(A, B, \cdots)$ and suppose that λ is preserved (in an obvious sense) when (1) is applied simultaneously to T and S. Such a mapping λ is called a natural equivalence. A large part of the paper is devoted to showing that various one-to-one correspondences arising in a natural manner in group theory, homology theory and so on are indeed "natural" in the sense that they can be regarded as natural equivalences between functors. These special cases serve also to illustrate various general theorems about functors and operations on functors.

P. A. Smith (New York, N. Y.).

Krishnan, V. S. The theory of homomorphisms and congruences for partially ordered sets. Proc. Indian Acad. Sci., Sect. A. **22**, 1–19 (1945). [MF 13397]

The author first recalls the properties of homomorphisms and congruence relations in general algebraic systems. He then lets $K, K', \cdots$ be systems with a commutative, associative, idempotent operation $\cup$ and zeros $0, 0', \cdots$. An "ideal" is a subset C of K which contains 0, contains $a\cup b$ if it contains a and b, and contains c if $c\cup x=a$ for some $a\in C$, $x\in K$. He shows that each ideal C determines a least congruence relation, which he calls regular; he also calls the associated homomorphism regular. He calls a homomorphism θ from K to K' irreducible if the only antecedent of $0'\in K'$ is $0\in K$. He shows that any homomorphism from K to K' is uniquely the product of a regular and an irreducible homomorphism; that any antecedent of a comple-

mented lattice under an irreducible lattice-homomorphism [terminology as in G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., vol. 25, New York, 1940; these Rev. 1, 325] is complemented; that a lattice-homomorphism from a modular lattice with 0, 1 to a complemented modular lattice is necessarily regular. He also obtains a new characterization of pseudo-complemented distributive lattices. *G. Birkhoff* (Cambridge, Mass.).

Krishnan, V. S. Homomorphisms and congruences in general algebra. Math. Student **13**, 1–9 (1945). [MF 14488]

An expository article. For details, cf. the preceding review. *G. Birkhoff* (Cambridge, Mass.).

Dieudonné, Jean. Sur la théorie de la divisibilité. Bull. Soc. Math. France **69**, 133–144 (1941). [MF 13242]

Let G be a partially ordered group, that is, a commutative group in which a partial ordering is defined such that $x>y$ implies $zx>zy$. It is proved that G can be imbedded in a partially ordered group which is a lattice if and only if $x^n\geqq 1$ implies $x\geqq 1$ for any $x\in G$ and any positive integer n. This result has been proved by P. Lorenzen [Math. Z. **45**, 533–553 (1939); these Rev. **1**, 101], but the present proof is direct; it shows, in fact, that G can be imbedded in a direct product of simply ordered groups. If A is an integral domain and G is the multiplicative group of the quotient field of A modulo the units of A, then the above condition on G means that $x^n\in A$ implies $x\in A$ for x in the quotient field of A; an example is given to show that this condition is weaker than integral closure. A further example shows that the finitely generated v-ideals of A need not form a group even if the cancellation law holds; this settles a question left open by Prüfer. *I. S. Cohen.*

Smiley, Malcolm F. A remark on metric Boolean rings. Bull. Amer. Math. Soc. **51**, 378–380 (1945). [MF 12514]

The author proves that a ring R is a metric Boolean ring if (and only if) it possesses a functional satisfying (i) $\mu(a)>0$ if $a\neq 0$ and (ii) $\mu(a+b)+2\mu(ab)=\mu(a)+\mu(b)$. (The factor 2 in (ii) occurs because R is a Boolean ring and not a Boolean algebra.) If (i) is weakened to (i′) $\mu(a)\geqq 0$, we get a Boolean ring modulo the ideal of elements for which $\mu(a)=0$. The referee remarked that $\mu(a)$ could be in any ring with unity if further assumptions were made on R. *G. Birkhoff* (Cambridge, Mass.).

Moisil, Gr. C. Sur les anneaux de caractéristique 2 ou 3 et leurs applications. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **12**, 66–90 (1941). [MF 13555]

The author introduces an algebra, based on the three-valued logic of Łukasiewicz [J. Łukasiewicz, C. R. Soc. Sci. Varsovie **23**, 51–77 (1930); J. Łukasiewicz and A. Tarski, ibid. **23**, 30–50 (1930)], which is analogous to the Boolean algebra based on classical logic. This leads to a study of commutative 3-rings, that is, rings such that $a^3=a$ for every element a. A 3-ring which is not a Boolean ring has characteristic 3 or 6. If the characteristic is 3, it is isomorphic to a subring of a direct sum of three-element fields [N. H. McCoy and D. Montgomery, Duke Math. J. **3**, 455–459 (1937)]; if of characteristic 6 it is a direct sum of a 3-ring of characteristic 3 and a Boolean ring. The author emphasizes the logical aspects of the theory, and also includes a short exposition of Stone's theory of Boolean rings.
N. H. McCoy (Northampton, Mass.).

Almeida Costa, A. On semiprimary rings. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. **14**, 38 pp. (1945) = Anais Fac. Ci. Pôrto **29**, no. 4. (Portuguese. German summary) [MF 13299]

The topics treated center about the nilpotence of the radical and the structure of a ring S relative to its radical. In part A, two versions of the radical are considered: R, the union of the nilpotent ideals, and R^*, the radical in the sense of Köthe. Levitzki's recent criterion for nilpotence of R [Duke Math. J. **11**, 367–368 (1944); these Rev. **6**, 34] is given, and various conditions for the existence of R^* are established. Part B gives various connections between idempotents in S and S/R^*. Part C is devoted to the structure of semiprimary rings (those for which R^* exists and S/R^* is semisimple). Results due to Köthe [Math. Z. **32**, 161–186 (1930)] are amplified. In part D, rings with minimum condition on right ideals are treated, following Hopkins [Ann. of Math. (2) **40**, 712–730 (1939); these Rev. **1**, 2] and Dieudonné [J. Reine Angew. Math. **184**, 178–192 (1942); these Rev. **5**, 32]. The author concludes with a proof of the existence of a splitting field of a separable algebra. *I. Kaplansky* (Chicago, Ill.).

Jacobson, N. A topology for the set of primitive ideals in an arbitrary ring. Proc. Nat. Acad. Sci. U. S. A. **31**, 333–338 (1945). [MF 13717]

A (two-sided) ideal $\mathfrak{J}$ in a ring $\mathfrak{A}$ is called primitive if 0 is the only $x\in\mathfrak{A}$ such that $\mathfrak{A}x\subset\mathfrak{J}$. Let S be the set of primitive ideals in $\mathfrak{A}$; a topology is introduced in S by defining the closure of a subset M of S to be the set of primitive ideals which contain the intersection of all ideals in M. The space defined in this way is a T_0-space, but generally not T_1 (except when $\mathfrak{A}$ is commutative). If $\mathfrak{A}$ has a unit element, S is compact (that is, bicompact). The space S is not changed if $\mathfrak{A}$ is replaced by $\mathfrak{A}/\mathfrak{R}$, where $\mathfrak{R}$ is the radical of $\mathfrak{A}$. If $\mathfrak{A}$ is semisimple and has a unit element, a necessary and sufficient condition for S to be disconnected is that there should exist a decomposition of $\mathfrak{A}$ as the direct sum of two two-sided ideals not equal to $\{0\}$. It is shown that, given any totally disconnected compact space S, it is possible to construct a ring $\mathfrak{A}$ whose space of primitive ideals is homeomorphic with S; the elements of S are certain mappings of S into a field (not necessarily commutative) which can be selected arbitrarily. *C. Chevalley.*

Gurevich, G. B. On some arithmetical invariants of an arbitrary matric Lie algebra. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 47–49 (1944). [MF 12570]

Let a Lie algebra $\mathfrak{A}$ whose elements are matrices of degree n be called a "null-algebra" if every matrix of $\mathfrak{A}$ is nilpotent. If E is an n-dimensional vector space, every null-algebra $\mathfrak{A}$ determines a chain of subspaces $0\subset V_1\subset\cdots\subset V_k=E$ such that, for all $A\in\mathfrak{A}$, $A^kV_k=0$. If $\mathfrak{A}$ contains every matrix A for which $AV_k\subseteq V_{k-1}$, call $\mathfrak{A}$ a "complete null-algebra." Every complete null-algebra is determined by the set of numbers $(r_1, r_2, \cdots, r_k)$, where r_k is the rank of V_k: these numbers the author calls the "arithmetical characteristics" of $\mathfrak{A}$. Let $\mathfrak{M}$ be any linear system of matrices and let $\mathfrak{S}=\mathfrak{S}(\mathfrak{M})$ be the set of all matrices S for which $SM-MS\in\mathfrak{M}$ for all $M\in\mathfrak{M}$; let $\mathfrak{R}=\mathfrak{R}(\mathfrak{M})$ be the set of all matrices R such that $\mathrm{tr}(SR)=0$ for all $S\in\mathfrak{S}$. Then both $\mathfrak{S}$ and $\mathfrak{R}$ are Lie algebras, and $\mathfrak{R}$ is a solvable ideal of $\mathfrak{S}$. The author states without proof several properties of the algebras $\mathfrak{S}$ and $\mathfrak{R}$ associated with the given set $\mathfrak{M}$. Among them are: (i) $\mathfrak{R}$ is a null-algebra; (ii) $\mathfrak{M}$ is a null-algebra if and only if $\mathfrak{M}\subseteq\mathfrak{R}$; (iii) $\mathfrak{M}$ is a complete null-algebra if and only if $\mathfrak{M}=\mathfrak{R}$;

(iv) a matric Lie algebra $\mathfrak{A}$ is completely reducible if and only if $\mathfrak{R}(\mathfrak{A})=0$. A method of associating a complete null-algebra with any linear system of matrices is also described. No mention is made of the coefficient field, but the reviewer suspects that the author's methods of proof are not rational.
S. A. Jennings (Vancouver, B. C.).

Gourevitch, G. Systèmes complets de bivecteurs. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 363–364 (1944). [MF 13544]

This is a brief report on further studies along lines indicated in the paper reviewed above. Let V be a linear system of "bivectors" v_{ij}, $i, j=1, 2, \cdots, n$, and let A be the set of matrices (A_i^α) such that $A_{i\alpha}^{\ \beta} v_{j\beta} \varepsilon V$ for all $v_{ij} \varepsilon V$, where the square brackets indicate the alternating product. It follows that A is a Lie algebra, which the author calls the Lie algebra of the linear system V. If A is the normalisor of a complete null-algebra B, then the system V is called a "complete system of bivectors" and a characterization of such systems in terms of the basis vectors of B is announced. Arithmetical characteristics are defined for a complete system of bivectors similar to those of a complete null-algebra, and relations between the two types of characteristics are discussed. *S. A. Jennings* (Vancouver, B. C.).

***Loonstra, Frans. Analytische Untersuchungen über Bewertete Körper.** Thesis, University of Amsterdam, 1941. xvi+101 pp.

Vorliegende Dissertation ist zum grossen Teil ein wörtlicher Abdruck einiger Noten des Verfassers [Nederl. Akad. Wetensch., Proc. **44**, 286–297, 397–408, 409–419, 568–576, 577–589, 700–710, 806–813 (1941); diese Rev. **3**, 103, 104, 264], die zusammen Kap. II–V der Dissertation bilden. Kap. I ist eine allgemeine Einleitung, Kap. VI eine Darstellung der Schnirelmannschen Untersuchungen über Funktionen in algebraisch abgeschlossenen bewerteten Körpern [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1938**, 487–498 (1938)]. *H. Freudenthal.*

Dieudonné, Jean. Sur les fonctions continues p-adiques. Bull. Sci. Math. (2) **68**, 79–95 (1944). [MF 13253]

The author extends a number of classical properties of continuous numerical functions to the case of functions taking values in a p-adic field: Urysohn extension theorem, approximation of a function $f(x, y)$ by a sum of terms of the form $\varphi(x)\psi(y)$, approximation of a function by polynomials, existence theorem for differential equations.
C. Chevalley (Princeton, N. J.).

Fueter, Rud. Abelsche Gleichungen in algebraischen Zahlkörpern. Comment. Math. Helv. **17**, 108–127 (1945).

The author considers an extension field K, cyclic of odd prime degree l over the algebraic number field k, with the restriction that k does not contain the lth roots of unity. He gives a new proof that the number of genera is at most $l^{h(l)-1}$, where $l^{h(l)}$ is the class number of k-ideals modulo principal k-ideals generated by norm residues. (Genera are ideal classes of K modulo the subgroup of all K-ideals whose norms to k are principal k-ideals generated by norm residues.) By use of the familiar analytically proved lower bound for the number of genera, he proves that the principal genus consists exactly of the $(1-S)$th powers of the ordinary ideal classes of K, where S is a generating automorphism for K over k. He also proves that every k-unit which is a norm residue is a norm.

The principal tool in the proof is the construction of a canonical basis for the ideal-complexes in K. Complexes are defined to be the classes of K-ideals modulo the subgroup generated by all principal K-ideals together with all k-ideals; the basis is similar to the familiar basis for a subgroup of the K-units. The proofs are somewhat complicated since the author avoids all the new methods, such as $\mathfrak{p}$-adic number theory, Herbrand's lemma and Hasse's group-stenographic notation, which have been introduced since the original paper of Takagi [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **41**, no. 9 (1920)]. *G. Whaples.*

Rédei, Ladislaus. Über die Klassengruppen und Klassenkörper algebraischer Zahlkörper. J. Reine Angew. Math. **186**, 80–90 (1944). [MF 13408]

The author studies the structure of an ideal class group $\mathfrak{H}$, of prime power order l^r, over an algebraic number field k, and its relation to the cyclic subfields of the class field to $\mathfrak{H}$. He constructs a special basis for the set of elements of period l in $\mathfrak{H}$, together with a set of cyclic subfields of degree l which has a certain relation to this basis. The invariants of the group $\mathfrak{H}$ are expressed in terms of this construction. The method is entirely group-theoretical, and the results could have been stated entirely in terms of the Galois theory of an Abelian extension field. This paper is related to earlier work of the author [same J. **180**, 1–43 (1938)] which discussed in much more detail the case when k is quadratic over the rational field and $l=2$.
G. Whaples (Philadelphia, Pa.).

Artin, Emil, and Whaples, George. Axiomatic characterization of fields by the product formula for valuations. Bull. Amer. Math. Soc. **51**, 469–492 (1945). [MF 12680]

Artin proved [J. Reine Angew. Math. **167**, 157–159 (1932)] that for algebraic number fields all valuations are generated by the finite and infinite prime ideals of the field. On the other hand, in a general field k the sets of equivalent and nontrivial valuations can be used to define a prime divisor of the field, thus generalizing also the valuations in function fields. The trivial valuation gives each nonzero element the value 1; two valuations are called equivalent if one is a positive real power of the other. Any real positive power of a valuation is here called a valuation even if it ceases to satisfy the axioms. By the valuation $|\alpha|_\mathfrak{p}$ of α a fixed element of the prime divisor $\mathfrak{p}$ is meant. No product relation exists among a finite number of nontrivial and inequivalent valuations in a field. A relation can, however, hold for an infinite number of valuations. This is the case for the algebraic number fields and algebraic extensions of a field of functions of one variable over a field of constants. It is shown that this property together with another characterize these two types of fields. More explicitly, the following two axioms characterize these fields. (1) There is a set $\mathfrak{M}$ of prime divisors $\mathfrak{p}$ and a fixed set of valuations $|\alpha|_\mathfrak{p}$, one for each $\mathfrak{p} \varepsilon \mathfrak{M}$, such that, for every $\alpha \neq 0$ of k, $|\alpha|_\mathfrak{p}=1$ for all but a finite number of $\mathfrak{p} \varepsilon \mathfrak{M}$ and

$$\prod_{\mathfrak{p} \varepsilon \mathfrak{M}} |\alpha|_\mathfrak{p} = 1.$$

(2) The set $\mathfrak{M}$ of (1) contains at least one prime divisor which is either discrete, with a residue class field of finite order, or Archimedean, with a completed field which is either the real or the complex field. [The condition on the completed field could be omitted according to a theorem by Ostrowski [Acta Math. **41**, 271–284 (1918)].] By the residue class field of a non-Archimedean prime divisor is under-

stood the residue class field of the ring of all elements of value less than or equal to 1 with respect to the elements of value less than 1. The order is said to be finite if the number of linearly independent elements with respect to the field of constants k_0 of k is finite if k has such a field, otherwise if the number of elements is finite.

The importance of this characterisation of two types of fields lies in the fact that the same fields are the ones for which the theorems of class field theory hold, provided that in the case of function fields the field of constants is assumed a Galois field. The authors conclude that these theorems must therefore be derivable directly from axiom 1 and a modified axiom 2. They carry this out, for instance, for the generalized Dirichlet unit theorem [see C. Chevalley, Ann. of Math. (2) 41, 394–418 (1940); these Rev. 2, 38] and the theorem that the class number is finite. Through these proofs they are able to show (1) that class field theory for the two types of fields need not be treated separately, (2) that class field theory can be developed without the use of ideal theory and lattice theory. *O. Todd-Taussky.*

THEORY OF GROUPS

Miller, G. A. Illustrations and simple abstract proof of Sylow's theorem. Proc. Nat. Acad. Sci. U. S. A. 31, 166–170 (1945). [MF 12502]

This proof of Sylow's theorem should be compared with that given by Burnside [The Theory of Groups of Finite Order, 2d ed., Cambridge University Press, 1911, pp. 149–150]. *G. de B. Robinson* (Toronto, Ont.).

Frucht, Roberto. On certain invariants of finite groups. Publ. Inst. Mat. Univ. Nac. Litoral 5, 199–213 (1945). (Spanish. English summary) [MF 13730]

Two elements a and b of a group G are called conjugate if one of them can be transformed into the other by an inner automorphism of G, that is, if there exists an element c of G such that $c^{-1}ac=b$; two sets of elements $a_1, \cdots, a_m$ and $b_1, \cdots, b_m$ are conjugate if one set can be thus transformed into the other. The author defines the invariant $N_i(m)$ as the number of nonconjugate sets of m members that can be formed from the elements of a finite group G. Thus $N_i(1)$ is the number of classes of conjugate elements of G and, if g is the order of G, $N_i(g)=1$. Certain properties of these invariants are proved and it is pointed out that for a commutative group $N_i(m)=\binom{g}{m}$, so that these invariants can be regarded as a generalization of the binomial coefficients. Another set of invariants $N_o(m)$ is defined using outer instead of inner automorphisms; these are not trivial for commutative groups. A third set of invariants $P(m)$ is defined as the number of nonproportional sets; two sets a_i and b_i are called proportional if there exists an element c of G such that the set a_ic is identical with the set b_ic.

H. W. Brinkmann (Swarthmore, Pa.).

Finan, E. J. Cyclic subsets of a group. Duke Math. J. 12, 509–513 (1945). [MF 13519]

Let G be a finite group, a a given positive integer and λ an element of G; then the set of elements $\lambda_0=\lambda$, $\lambda_1=\lambda^a$, $\lambda_2=\lambda^{a^2}$, $\cdots$ is called a power set of G. If $\lambda_0=\lambda_n$ for some integer n, the set is called a cyclic subset of G of order n and power a, and is denoted by the symbol $J\lambda_{an}$. For any G the elementary properties of cyclic subsets are discussed, and among the results obtained are: (i) G contains a $J\lambda_{an}$ if and only if it contains an element λ of period e such that a belongs to n modulo e; (ii) all the elements of a $J\lambda_{an}$ have the same period, and any of them generates the set; (iii) if $J\lambda_{an}\neq J\mu_{am}$ then $J\lambda_{an}\cap J\mu_{am}=0$. The second half of the paper is devoted to the case when G is an additive cyclic group of order t. Let a and n be such that $(a,t)=1$ and a belongs to n mod t, and let $f(a,t)$ be the number of elements that fall into sets $J\lambda_{an}$. An explicit formula for the function $f(a,t)$ is given in terms of the divisors of t, and the properties of f are investigated. For example, $f(a,t)=\sum\phi(t_i)$, where the t_i are suitably chosen divisors of t; while if a is a primitive root mod p^r and mod q^s (p and q odd primes), then $f(a,p^r)=\phi(p^r)$ and $f(a,p^rq^s)=\phi(p^r)\phi(q^s)$.

S. A. Jennings (Vancouver, B. C.).

Rutherford, D. E. On substitutional equations. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 117–126 (1944). [MF 11545]

The study of substitutional equations of the form $LX=0$, where L is a given substitutional expression and X an unknown one, goes back to A. Young. Denoting the $n!$ permutations of the symmetric group S_n by $\epsilon, \sigma_2, \sigma_3, \cdots, \sigma_{n!}$, we may write

$$L=l_1\epsilon+l_2\sigma_2+\cdots+l_{n!}\sigma_{n!},$$

where the l_i are numerical. By expressing L and X in terms of the basis elements e^{α}_{rs} of the enveloping algebra of S_n we have

$$L=\sum_{\alpha,r,s} l^{\alpha}_{rs}e^{\alpha}_{rs}, \quad X=\sum_{\alpha,r,s} x^{\alpha}_{rs}e^{\alpha}_{rs},$$

where the number of basis elements corresponding to a partition α of n is f^{α}. Such expressions lead to matrix equations $U^{\alpha}_L U^{\alpha}_X=0$. If the rank of U^{α}_L is λ^{α}, then there are $f^{\alpha}-\lambda^{\alpha}$ independent parameters in the general solution for the elements of a given column of U^{α}_X, so that, in all, the number of linearly independent solutions of the equation $LX=0$ is $\sum_{\alpha}f^{\alpha}(f^{\alpha}-\lambda^{\alpha})$.

The novelty of the present paper consists in pointing out the significance of the case where L is idempotent. Under such circumstances the number of linearly independent solutions of the equation $LX=0$ is $(1-l_1)n!$, and the most general solution is given by $X=(\epsilon-L)Y$, where Y is an arbitrary substitutional expression. In certain special cases it is possible to isolate the linearly independent solutions; for example, if $L=H\equiv(1/h)(\epsilon+\tau_2+\cdots+\tau_h)$, where $\epsilon, \tau_2, \cdots, \tau_h$ form a subgroup of S_n of order h.

If L is not idempotent let $\varphi_L(x)=0$ be the minimum equation of L. It follows immediately that the only solution of $LX=0$ is $X=0$ unless $\varphi_L(x)$ has the factor x. The first step in the study of this general case is to show that $LX=0$ may be replaced by another equation $M(x)=0$ having the same solutions as $LX=0$ and such that $\varphi_M(x)$ has the factor x to the first power only. If the minimum equation of M is then $x\psi(x)=0$, it follows that $\psi(M)$ and also $\{\epsilon-\psi(M)\}$ are idempotent, so that the equations $MX=0$ and $\{\epsilon-\psi(M)\}X=0$ have the same solutions. From the preceding case, the number of linearly independent solutions of $Mx=0$ is $kn!$, where k is the coefficient of ϵ in $\psi(M)$ and the most general solution is given by $X=\psi(M)Y$, where Y is an arbitrary substitutional expression as before. The study of equations of the form $LX=R$ or $MX=S$ is now possible and a necessary condition that $MX=S$ have a solution is that $\psi(M)S=0$. Under such circumstances, if

$x\psi(x)=x(1-x\theta(x))$, the most general solution is given by $X=\theta(M)S+\psi(M)Y$, where Y again is arbitrary. Simultaneous equations do not present any new difficulties. The paper concludes with two examples worked out for the case $n=3$. *G. de B. Robinson* (Toronto, Ont.).

Gårding, Lars. A general theorem concerning group representations. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **13**, no. 24, 229–235 (1943). [MF 12591]

This note is concerned with a generalization of the familiar Burnside-Schur lemma of representation theory. With all coefficients in a fixed ground field $\mathbf{P}$, let $G=\{g\}$ be a group each of whose representations is the direct sum of irreducible representations (true for finite and certain infinite groups). Let $g\to S(g)$, $g\to T(g)$, $g\to A(g)=(a_{ij}(g))$, where $i,j=1,\cdots,q$, be any representations of G. The problem is: to construct all sets $U=(U_1,\cdots,U_q)$ of (in general non-square) matrices such that for all $g\in G$ and $j=1,\cdots,q$,

$$S(g)\supset U_jT^{-1}(g)=\sum_{i=1}^{q}\alpha_{ij}U_i.$$

This construction is effected by means of an explicit reduction of the Kronecker product representation $S\times T$ in terms of a primitive set of representations of G. *A. L. Foster*.

Venkatarayudu, T. Characters of the classes of the form (n_1, n_2, n_3) in symmetric groups. Proc. Indian Acad. Sci., Sect. A. **22**, 42–45 (1945). [MF 13399]

The paper contains further special numerical values. [Cf. J. Indian Math. Soc. (N.S.) **7**, 42–45 (1943); these Rev. **5**, 58.] *G. de B. Robinson* (Toronto, Ont.).

Comét, Stig. Une propriété des déterminants et son application au calcul des caractères des groupes symétriques. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 7, 84–94 (1945). [MF 13543]

The author studies the characters χ_κ^ρ of the irreducible representations of the symmetric group $\mathfrak{S}_N$ by applying the following theorem to certain determinants $\Delta(\tau)$ associated with $\mathfrak{S}_N$. Let $x_1,\cdots,x_m$ be a set of m independent variables whose elementary symmetric functions c_n are given by $\prod(x+x_i)=\sum c_nx^{m-n}$ and whose sums of powers are denoted by $s_n=\sum x_i^n$. Then if a_{jk} are arbitrary variables the author shows that $c_n|a_{jk}|=\sum|a_{jk}x_k^{e_j}|$, where the sum is extended over all the ${}_mC_n$ combinations of n rows $(r_1,\cdots,r_n)$ and $e_j=1$ for the n rows of the particular combination and $e_j=0$ for the other $m-n$ rows. Now let $\tau=(t_1,\cdots,t_m)$ be a properly decreasing set of m nonnegative integers, and let $r_i=t_i+i-m$. Then $\rho=(r_1,\cdots,r_m)$ will be a nonincreasing set of positive integers and zeros which define a partition ρ of the sum $N-n=\sum r_i$. (The letter π will denote a corresponding partition of N.) For convenience, let us call τ a proper m-ladder, call an even or odd permutation of the integers t_i in τ an evenly or oddly deranged m-ladder, and call $\iota=(m-1,m-2,\cdots,1,0)$ the unit m-ladder. If $\Delta(\tau)$ is the determinant $|x_i^{t_j}|$, then $\Delta(\iota)$ is the product of the differences of the x_i's, and the quotient $\Phi^\rho=\Delta(\tau)/\Delta(\iota)$ is a symmetric function depending only on the positive integers in the partition ρ and not on the length m which determines the number of 0's that follow. Schur has shown that the "characteristic function" Φ^ρ is equal to $\sum\chi_\kappa^\rho h_\kappa s_\kappa/(N-n)!$, where h_κ is the number of elements in the class κ of $\mathfrak{S}_{N-n}$ and $s_\kappa=s_{k_1}s_{k_2}\cdots s_{k_{N-n}}$. Furthermore, $s_\kappa=\sum_\rho\chi_\kappa^\rho\Phi^\rho$. The author's theorem on determinants gives immediately the expansion (*) $s_\sigma\Phi^\rho=\sum\epsilon_{\sigma\tau}^\rho\Phi^\tau$, where ρ is a fixed partition of $N-n$ and τ runs over all proper partitions of N. Here $\epsilon_{\sigma\tau}^\rho=+1$ or -1, respectively, if the addition of n to one of the integers in the m-ladder τ gives an evenly or oddly deranged m-ladder corresponding to π, but $\epsilon_{\sigma\tau}^\rho=0$ if this addition of n produces an improper m-ladder having two equal integers. For fixed n and ρ there are just N partitions π for which $\epsilon_{\sigma\tau}^\rho\neq0$. It is also proved that, if N is fixed and ρ and σ are two partitions of numbers less than N, then $\sum_\pi\epsilon_{\sigma\pi}^\rho\epsilon_{\sigma\pi}^\tau=N\delta_{\rho\tau}$. This yields an inverse to equations (*), namely $N\Phi^\tau=\sum_{\sigma,\rho}\epsilon_{\sigma\tau}^\rho s_\sigma\Phi^\rho$. With the aid of these formulas, the author derives an extension of Murnaghan's recurrence formula for characters, an explicit expression for χ_κ in terms of the matrices defined by the $\epsilon_{\sigma\tau}^\rho$, a simple factorial expression for χ_κ^ρ when $\rho=P^q$, $\kappa=q^p$, and reduction formulas involving the suppression of cycles of length n.

J. S. Frame (East Lansing, Mich.).

Levi, F. W. Notes on group-theory. IV–VI. J. Indian Math. Soc. (N.S.) **8**, 78–91 (1944). [MF 13271]

[Notes I–III appeared in the same J. (N.S.) **8**, 1–9, 44–56 (1944); these Rev. **6**, 40, 202.] If A and B are groups, the author defines $[A,B]$ as the group generated by elements $[a,b]$ for a in A and b in B, subject to the relations $[aa',b]=[a,b][a',b]$ and $[a,bb']=[a,b][a,b']$ for a,a' in A and b,b' in B. The operation $[a,b]$ constitutes a pairing operation of the groups A and B with values in $[A,B]$ which is most general in the sense that there exists to every pairing operation $\{a,b\}$ with values in a group V a homomorphism φ of $[A,B]$ into V such that $[a,b]^\varphi=\{a,b\}$. Thus all the fully invariant properties derived for the operation $[a,b]$ will be valid for every pairing operation $\{a,b\}$. The commutativity of the group $[A,B]$ is an example of such a fully invariant property. A property of $[A,B]$ which is not fully invariant is the fact that $[A,B]$ is the direct product of $[A',B]$ and $[A'',B]$ if A is the direct product of A' and A''.

This construction has many applications. If G is a group with Abelian central quotient group, then its commutator subgroup is a homomorphic image of $[G,G]$. Another instance may be obtained as follows. If $A*B$ is the free product of the groups A and B, denote by C the normal subgroup generated by all the commutators $(a,b)=a^{-1}b^{-1}ab$ for a in A and b in B, and denote by D the normal subgroup of $A*B$ which is generated by all the commutators $(a,(a',b))$ and $(b,(b',a))$ for a,a' in A and b,b' in B. Then C/D is essentially the same as $[A,B]$; and every element in the S-product $A\circ B=(A*B)/D$ of A and B may be represented in one and only one way in the form abc with a in A, b in B, c in $[A,B]$. A noteworthy property of these S-products is the identity $A\circ B=A'\circ B$ whenever A' is conjugate to A in $A\circ B$. *R. Baer* (Urbana, Ill.).

Zappa, Guido. Sul comportamente degli elementi periodici in un gruppo di Dedekind infinito. Comment. Math. Helv. **18**, 42–44 (1945).

The author considers groups G with the property that Dedekind's law is satisfied by the partially ordered set of all the subgroups of G. If G contains elements of infinite order, then the author shows that the set H of all the elements of finite order in G is a subgroup of G with the following property: every subgroup of H is a normal subgroup of G. *R. Baer* (Urbana, Ill.).

Jones, Alfred W. The lattice isomorphisms of certain finite groups. Duke Math. J. 12, 541–560 (1945). [MF 13523]

A group G is said to be modular if $L(G)$, the lattice of its subgroups, is modular; G is L-indecomposable if $L(G)$ is indecomposable; G and H are L-isomorphic if $L(G)$ is isomorphic to $L(H)$. In the first part of the paper the author obtains necessary and sufficient conditions that finite groups G and H be L-isomorphic; these criteria are group-theoretical in nature, and involve the generators and the orders of the maximal cyclic subgroups of G and H. Conditions for L-decomposability are then investigated. K. Iwasawa has shown [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 171–199 (1941); these Rev. 3, 193] that a sufficient condition for a group to be L-decomposable is that it is a direct product of groups of relatively prime orders; the author shows that this condition is also necessary, and hence that the only L-indecomposable modular groups are those of the three types described by Iwasawa. It is also shown that a decomposable modular group can be expressed as the union of a normal Abelian subgroup and a cyclic subgroup, each of suitable type. The remainder of the paper is devoted to a discussion of L-isomorphic modular groups. Necessary and sufficient conditions are obtained for L-isomorphism of modular groups, and the problem of when L-isomorphism implies isomorphism is solved completely for groups of this type. *S. A. Jennings* (Vancouver, B. C.).

Braconnier, Jean. Groupes d'automorphismes d'un groupe localement compact. C. R. Acad. Sci. Paris 220, 382–384 (1945). [MF 13951]

The author defines a topology in the group $\mathfrak{G}(G)$ of automorphisms of a locally compact group G; the familiar operations are continuous in this topology. The same topology is used in the Galois group $\mathfrak{G}(K, K')$, where K is a locally compact field and K' is a closed subfield. If K is discrete and commutative, $\mathfrak{G}(K, K')$ is compact if and only if K is a separable normal algebraic extension of K'. If u is an automorphism of G, its modulus $\Delta(u)$ is the positive number defined by $\int f(u^{-1}(x))dx = \Delta(u)\int f(x)dx$, where integration is in the sense of Haar and f is any integrable function on G; $\Delta(u)$ is a multiplicative homomorphism of $\mathfrak{G}(G)$ into the positive numbers. It is indicated how Δ may be used to obtain in a simple manner the classical theorems on the structure of locally compact fields.

N. E. Steenrod (Ann Arbor, Mich.).

Braconnier, Jean, et Dieudonné, Jean. Sur les groupes abéliens localement compacts. C. R. Acad. Sci. Paris 218, 577–579 (1944). [MF 13468]

The authors state extensions of the results of the note reviewed above to the case of locally compact Abelian groups. To obtain these, they introduce the notion of the direct product of a family of locally compact groups modulo a family of open subgroups. Neither the resulting group nor its topology is quite that of the ordinary product.

N. E. Steenrod (Ann Arbor, Mich.).

Braconnier, Jean. Sur les groupes topologiques primaires. C. R. Acad. Sci. Paris 218, 304–305 (1944). [MF 13394]

A topological group G is primary (relative to the prime p) if for each x in G the homomorphism $n \to x^n$ of the integers Z into G admits a continuous extension to the group of p-adic integers Z_p. Various properties of primary groups are stated. For locally compact Abelian groups, the character group of a primary group relative to p is another primary group relative to p. The only compact primary Abelian groups in which every element is of infinite order are the direct products of groups isomorphic to Z_p. If "infinite" is replaced by "finite," the only such groups are direct products of cyclic groups whose orders are powers of p and are bounded. *N. E. Steenrod* (Ann Arbor, Mich.).

Montgomery, Deane. What is a topological group? Amer. Math. Monthly 52, 302–307 (1945). [MF 12499]

An expository article. *G. Birkhoff.*

Montgomery, Deane. Compact groups of transformations each of which is analytic. Ann. of Math. (2) 46, 365–371 (1945). [MF 13424]

Bochner, S. Compact groups of differentiable transformations. Ann. of Math. (2) 46, 372–381 (1945). [MF 13425]

Montgomery, Deane. Topological groups of differentiable transformations. Ann. of Math. (2) 46, 382–387 (1945). [MF 13426]

Quoiqu'on connaisse bien les groupes transitifs compacts, la connaissance des groupes intransitifs est toujours imparfaite. Quand on fait abstraction de théorèmes plus spéciaux [B. de Kerékjártó, voir par ex. Acta Math. 74, 129–173 (1941); ces Rev. 4, 3; D. Montgomery et L. Zippin, Trans. Amer. Math. Soc. 40, 24–36 (1936); Bull. Amer. Math. Soc. 46, 520–521 (1940); ces Rev. 2, 6; S. B. Myers et N. E. Steenrod, Ann. of Math. (2) 40, 400–419 (1939)], on doit se borner aux recherches de M. H. Cartan [Sur les Groupes de Transformations Analytiques, Actual. Sci. Ind. no. 198, Hermann, Paris, 1935]. Cependant M. Cartan a déjà esquissé la démonstration d'un théorème valable pour des groupes (même locaux) constituées par des transformations à dérivées partielles simultanement continues au voisinage du point considéré: ces groupes ne contiennent pas de sous-groupes arbitrairement petits. Ce théorème repose sur le fait analytique que l'écart d'une transformation d'un tel groupe sera approximativement multiplié avec p quand on élève cette transformation à la p-ième puissance.

Le but des trois mémoires que nous allons analyser est de renforcer et de généraliser les résultats de M. Cartan sur les groupes de transformations analytiques complexes. Le théorème final de M. Montgomery est le suivant. Un groupe compact opérant effectivement par des transformations de classe C^1 (première note: transformations analytiques) dans une variété de classe C^1 (analytique) ne contient pas de sous-groupes arbitrairement petits, c'est-à-dire il est un groupe de Lie, considéré du point de vue abstrait topologique. M. Bochner, dans son théorème correspondant, suppose (comme M. Cartan) les transformations du groupe uniformement de la classe C^1, mais au cours de sa démonstration il se sert de l'intégration de Haar, c'est-à-dire d'une ressource moins élementaire et plus globale, et qui n'est applicable qu'aux groupes compacts.

M. Montgomery, qui, dans sa première communication, a également fait usage de cet instrument, a réussi à s'en libérer et est parvenu dans sa dernière note au lemme suivant, valable pour tous les groupes localement compacts de transformations de classe C^1. Les transformations $f(x, a)$ d'un tel groupe ont leurs dérivées partielles du premier ordre $\partial f/\partial x$ simultanement continues dans les x et a. Ce théorème, qui (en vertu du théorème de M. Cartan) conduit immédiatement au théorème final énoncé plus haut, peut être employé pour affaiblir les conditions sous lesquelles M. Bochner énonce quelques théorèmes de sa note.

M. Bochner traite les familles compactifiables C^k (c'est-à-dire douées de dérivées partielles d'ordre k bornées et

uniformement continues dans chaque ensemble d'un système de recouvrement). Il démontre que si une telle famille est un groupe compact, elle peut être linéarisée dans l'entourage d'un point invariant au moyen d'une transformation C^0. De ce théorème il dérive la non-existence de sous-groupes arbitrairement petits. Quant aux familles générales compactes C^k, il démontre par exemple que si une telle famille contient l'identité, chaque transformation, qui n'en diffère pas trop, possède une inverse locale. Si T est une transformation d'un domaine D en elle-même, si la famille des itérées T^n est compactifiable C, si chaque ensemble compact de D est transformé en un ensemble compact, uniformément pour tous les T^n, et si en outre il y a un point invariant où le jacobien est ± 1, la transformation T est une automorphie de D. *H. Freudenthal* (Amsterdam).

Stiefel, E. Kristallographische Bestimmung der Charaktere der geschlossenen Lie'schen Gruppen. Comment. Math. Helv. 17, 165–200 (1945).

With every connected semisimple compact Lie group G there is associated a "crystallographic" transformation group Γ operating in Euclidean space R^l and generated by reflections across hyperplanes ("crystallographic" means that Γ contains a discrete translation subgroup of rank l). This association comes from taking R^l to be the minimal covering space of a maximal toroidal subgroup T of G; Γ is induced by the finite group of inner automorphisms of G which transform T into itself [see an earlier paper by the author, same Comment. 14, 350–380 (1942); these Rev. 4, 134]. If G is understood to be simply connected, the correspondence between the groups G and the crystallographic groups Γ generated by reflections is one-to-one. The algebraic and geometric properties of G are theoretically obtainable from knowledge of the corresponding Γ. In the present paper the author obtains from Γ the characters of the irreducible representations of G. A character of G is of course completely determined by its values on T and can therefore be taken to be a function of $y_1, \cdots, y_l$, where the y's are a coordinate system for R. If this coordinate system is properly chosen, the characters can be given explicitly by a function of $y_1, \cdots, y_l$ and of parameters $q_1, \cdots, q_l$ taking arbitrary integral values. The function is formed with the aid of the operations of Γ and involves in an explicit manner the linear forms which define the hyperplanes of the reflections which define Γ. *P. A. Smith.*

Malcev, A. On linear Lie groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 87–89 (1943). [MF 11169]

A Lie group is said to be linear if it admits an isomorphic matric representation. The author sketches proofs of the following theorems. (1) A connected Lie group is linear if and only if its radical and maximal semisimple subgroups are linear. (2) If L is a connected semisimple linear Lie group and Z is a discrete invariant subgroup of L, then L/Z is linear. (3) A connected solvable group is linear if and only if it is the semidirect product of its maximal compact subgroup by a simply connected invariant subgroup (if $a \to \varphi_a$ is a homomorphism of a group A into the group of automorphisms of another group B, then the set of pairs (a, b) with $a \in A$ and $b \in B$, with multiplication defined by $(a_1, b_1)(a_2, b_2) = (a_1 a_2, \varphi_{a_2}(b_1) b_2)$, is called a semidirect product of A by B). *G. W. Whitehead* (Princeton, N. J.).

Malcev, A. On the theory of the Lie groups in the large. Rec. Math. [Mat. Sbornik] N.S. 16(58), 163–190 (1945). (English. Russian summary) [MF 13008]

The main questions treated are the topological structure of Lie groups and the condition for the subgroup of a Lie group determined by a subalgebra of a Lie algebra to be closed. The main tool is the consideration of the maximal compact subgroups of a Lie group $\mathfrak{G}$; theorem 7 states that all these maximal connected compact subgroups are conjugate to each other. Theorem 11 states that any connected Lie group is homeomorphic to the product of any one of its maximal compact subgroups by a Euclidean space. This theorem was known for solvable groups and for the adjoint groups of semisimple groups. The author gives a new proof for the case of solvable groups [while writing his paper he was unaware of the existence of a former proof by the reviewer] and then extends the result to the case of arbitrary groups by making use of Levi's theorem.

Let G be the Lie algebra of a group $\mathfrak{G}$ and let H be a subalgebra of G; then there corresponds to H a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, which is a Lie group but which is not necessarily a closed subgroup of $\mathfrak{G}$. The author proves a number of results about the relationship between $\mathfrak{H}$ and its closure $\bar{\mathfrak{H}}$, notably the following: the Lie algebra of $\bar{\mathfrak{H}}$ is the direct sum of H and of a commutative algebra. He shows that each of the following conditions is necessary and sufficient for $\mathfrak{H}$ to be closed: (a) the closure of any one-parameter subgroup of $\mathfrak{H}$ is in $\mathfrak{H}$; (b) the intersection of $\mathfrak{H}$ with any compact subgroup of $\mathfrak{G}$ is closed. Making use of these conditions, he finds an algebraic criterion for $\mathfrak{H}$ to be closed which involves H and a certain modulus over the field of rational numbers which depends on $\mathfrak{G}$.

Unfortunately, the proofs contain several errors, some of which can be corrected easily while others seem much deeper. To the first category belong the insufficient proof of lemma 2 and the statement that "the closure of a one-dimensional nonclosed subgroup is compact" (p. 186, twice). To the second category belong the omission of the condition that $\mathfrak{G}$ be simply connected in theorems 3 and 8 (the second, third and fourth statements of theorem 3 are false in general, and therefore theorem 8 is also false in general) and the statement contained in the proof of theorem 11 to the effect that "the centrum of $\mathfrak{L}$ is contained in $\mathfrak{C}$" (false if the centrum of $\mathfrak{L}$ is infinite, which happens for instance if $\mathfrak{L}$ is the simply connected covering group of the real fractional linear group). Besides these errors, it should be mentioned that the proof of the all-important theorem 7 is almost unintelligible in several places. *C. Chevalley.*

ANALYSIS

Ciorănescu, Nicolas. Quelques formules de moyenne et quelques inégalités entre les valeurs moyennes des fonctions monotones. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 37–40 (1941). [MF 13554]

Let $f(x)$ and $g(x)$ have continuous derivatives in (a, b) and let $p(x)$ be nonnegative. Writing $M_p[\varphi] = P^{-1}\int_a^b p(x)\varphi(x)dx$, $P = \int_a^b p(x)dx$, the author proves that there are numbers ξ_1, ξ_2 in (a, b) such that

$$M_p[fg] - M_p[f]M_p[g] = \tfrac{1}{2}P^{-2}f'(\xi_1)g'(\xi_2)\int_a^b\int_a^b (v-u)^2 p(u)p(v)du\,dv.$$

If f and g are monotonic, this implies a weak form of an inequality of Chebyshev [see, for example, Hardy, Little-

wood and Pólya, Inequalities, Cambridge University Press, 1934, p. 168, theorem 236]. Various generalizations are given. *R. P. Boas, Jr.* (Providence, R. I.).

Sircar, H. On the inequality satisfied by the derivative of order n of a function. Bull. Calcutta Math. Soc. **37**, 21–23 (1945). [MF 13304]

Let $\{\varphi_n(x)\}$ be an orthonormal set on (a, b) of the form $\{\psi(x)F_n^{(n)}(x)\}$, and let $f(x)$ be a function of class $C^n(a, b)$, where (i) $F_n^{(p)}(a)=F_n^{(p)}(b)=0$, $p=0, 1, \cdots, n-1$; (ii) $F_n(x)$ does not change sign in (a, b); (iii) $f(x)/\psi(x)\epsilon L^2(a, b)$. Then

$$(1)\quad \beta_n=\inf_{a\leqq x\leqq b}|f^{(n)}(x)|\leqq\left\{\int_a^b|f/\psi|^2dx\right\}^{\frac{1}{2}}\left|\int_a^b F_n(x)dx\right|^{-1}.$$

For example, if the $\varphi_n(x)$ are the normalized Chebyshev polynomials, (1) gives

$$\beta_n\leqq(2n)!2^{-n-\frac{1}{2}}(\Gamma(n+\tfrac{1}{2}))^{-1}\left\{\int_{-1}^1 f^2(x)(1-x^2)^{-\frac{1}{2}}dx\right\}^{\frac{1}{2}},$$

when the integral on the right exists.

Nothing is said about "best possible" results. Certainly the estimates have no content if $f^{(n)}(x)$ has any zeros. Conditions (i)–(iii) are not those given by the author, whose hypotheses are more stringent than necessary; in fact, too stringent for applications he makes. *H. Pollard.*

Aronszajn, N., and Hardy, G. H. Properties of a class of double integrals. Ann. of Math. (2) **46**, 220–241 (1945). [MF 12399]

This paper originated in a problem concerning harmonic functions, to be published separately, which was first solved by reducing it to the problem on definite integrals which is the subject of the present paper. The theorems here given, though no longer needed for the solution of the original problem, are still useful for its extensions. The integral problem is as follows. Suppose that $\kappa(\xi, \eta)$ is measurable, positive and finite (almost everywhere) for positive ξ, η; that it is homogeneous of degree $2r-2$; and that

$$\int_0^\infty\kappa(1, \zeta)d\zeta=\rho_1<\infty,\quad \int_0^\infty\kappa(\zeta, 1)d\zeta=\rho_2<\infty.$$

Write

$$K(\chi)=\int_0^\infty\int_0^\infty\kappa(\xi, \eta)|\chi(\xi, \eta)|^2d\xi d\eta$$

and denote by $\mathbf{X}$ the class of χ for which $K(\chi)<\infty$. Denote by Φ, Ψ, Δ the subclasses of $\mathbf{X}$ in which χ has the special forms

$$\text{(i)}\quad \chi=\varphi(\xi),\qquad \text{(ii)}\quad \chi=\psi(\eta),\qquad \text{(iii)}\quad \chi=\varphi(\xi)-\psi(\eta).$$

Clearly $\varphi-\psi$ is Δ if φ is Φ and ψ is Ψ. The converse is not true, for if γ is constant $K(\gamma)=\infty$. It follows (using Minkowski's inequality) that $\varphi-\gamma$ can be Φ for at most one γ. The main problem is, given a $\varphi-\psi$ of Δ, to find, if possible, a (constant) γ such that $\varphi-\gamma$ is Φ and $\psi-\gamma$ is Ψ. It is found that this is possible when $r\neq 0$, but not always when $r=0$.

By simple transformations the problem is simplified as follows. Putting $\xi=e^x$, $\eta=e^y$, $\zeta=e^z$, $\varphi(\xi)=u(x)$, $\psi(\eta)=v(y)$, $\chi(\xi, \eta)=w(x, y)$, $\kappa(\xi, \eta)=e^{(r-1)(x+y)}k(y-x)$, we have

$$\rho_1=\int_{-\infty}^\infty e^{rz}k(z)dz<\infty,\quad \rho_2=\int_{-\infty}^\infty e^{rz}k(-z)dz<\infty,$$

$$K(w)=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{(x+y)r}k(y-x)|w(x, y)|^2dxdy.$$

Define K as the class of w for which $K(w)<\infty$, and U, V, D as the subclasses of K in which w has the special forms $u(x)$, $v(y)$, $u(x)-v(y)$, respectively. The problem now is, given a $u(x)-v(y)$ of D, to find, if possible, a constant c such that $u-c$ is U, $v-c$ is V. The main theorems are as follows. (1) If $r\neq 0$ and $u(x)-v(y)$ is D then there is a c such that $u(x)-c$ is U and $v(y)-c$ is V. (2) The conclusion of theorem (1) is not always true when $r=0$: given k, we can find u and v so that $u-v$ is D while $u-c$, $v-c$ are not U, V, respectively, for any c.

Three proofs of theorem (1) are given. The first two involve the direct use of theorems on convergence in mean; in the third the Plancherel theory of Fourier transforms is used. In a later section the theorems are restated and discussed in the language of Hilbert space. Finally, generalizations are given which correspond to the passage from L^2 to L^p. *H. E. Bray* (Houston, Tex.).

Beckenbach, E. F., and Bing, R. H. On generalized convex functions. Trans. Amer. Math. Soc. **58**, 220–230 (1945). [MF 13309]

To test a curve $y=f(x)$ for ordinary convexity, we construct the chord determined by (x_1, y_1) and (x_2, y_2) and compare the ordinate of the chord with that of the curve at $(x_1+x_2)/2$. The functions considered here are tested similarly, but with the chords replaced by an arbitrary family of continuous curves such that a unique member joins any two points. The principal results show that, for the generalized convex functions, boundedness implies continuity. *P. Franklin* (Cambridge, Mass.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur. IX. Inégalités linéaires et bilinéaires entre les fonctions convexes. Quelques généralisations d'une inégalité de Tchebycheff. Bull. Math. Soc. Roumaine Sci. **43**, 85–141 (1941). [MF 12731]

[Note VIII appeared in the same Bull. **22**, 34–41 (1939); these Rev. **1**, 204.] A function is convex of order n in an interval if all its differences of order $n+1$ are positive there. Let

$$A(f)=\int_a^b f(x)d\alpha(x),$$

$\alpha(x)$ of bounded variation [the author considers a somewhat more general functional]. Then (I) $A(f)>0$ for every function which is convex of order n in $[a, b]$ if and only if (1) $A(x^i)=0$ $(i=0, 1, \cdots, n)$, (2) $A(\varphi_{n+1,\lambda})=0$ $(a<\lambda\leqq b)$, where $\varphi_{n+1,\lambda}$ is zero in $[a, \lambda]$ and equal to $(x-\lambda)^n$ in $[\lambda, b]$, and (3) $A(x^{n+1})>0$; $A(f)\geqq 0$ for every function nonconcave of order n if and only if (1) and (2) are satisfied.

Let $s_k=A(x^k)$ and $\delta_k=|s_{i+j}|_{i,j=0}^k$, and let $P_k(x)$ be a polynomial of degree k for which $A(P_k(x)x^i)=0$ for $i=0, 1, \cdots, k-1$; the $P_k(x)$ are then the orthogonal polynomials with weight function $\alpha(x)$. (II) If A is a positive functional $(A(f)>0$ for every nonnegative continuous f with the exception of $f\equiv 0)$ and $\delta_{n+1}>0$, then $A(fP_{n+1})>0$ for every f which is convex of order n. Results generalizing (I) and (II) are proved for bilinear functionals. The proofs are given first for functions which are convex of order n on a discrete set of points, and then extended to the continuous case by approximating the general function which is convex of order n by functions of simple structure having the same property. (III) As consequences of these results, a number of inequalities are proved, for example, if A is a nonnegative functional and $\delta_{n+1}>0$, then $A(fg)>\sum_{r=0}^n A(fP_r)A(gP_r)$ if f and g are convex of order n.

(IV) Let $A(1)=0$, and let $M_k=\sup|A(f)|$ for all $f(x)$ which are k-fold monotonic and satisfy $0\leqq f(x)\leqq 1$; let M^* be defined similarly for completely monotonic functions. For bilinear functionals $A(f,g)$, the M_k are replaced by $N_k=\sup|A(f,f)|$. The author studies a number of properties of the sequences $\{M_k\}$ and $\{N_k\}$ and computes them for some particular cases. An interesting special case is

$$A(f,g)=\int_{-1}^{1}f(x)g(x)dx-\tfrac{1}{2}\int_{-1}^{1}f(x)dx\int_{-1}^{1}g(x)dx,$$

where $N_1=\frac{1}{2}$, $N_2=2/9$, $N_3=9/50$, $N_4=N_5=\cdots=N^*=8/45$ [Grüss, Math. Z. 39, 215–226 (1934); Landau, Prace Mat.-Fiz. 44, 337–351 (1937)]. Hardy [J. London Math. Soc. 11, 167–170 (1936)] obtained $N^*=1/6$ for an infinite interval. *R. P. Boas, Jr.* (Providence, R. I.).

Popoviciu, Tiberiu. Introduction à la théorie des différences divisées. Bull. Math. Soc. Roumaine Sci. 42, no. 1, 65–78 (1940). [MF 12739]

A collection of results, mostly well-known, concerning divided differences. *L. M. Milne-Thomson* (Greenwich).

Ganapathy Iyer, V. On singular functions. J. Indian Math. Soc. (N.S.) 8, 94–108 (1944). [MF 13273]

The author calls a function $f(x)$ singular in the closed interval $[a,b]$ if it is of class C^∞, not identically zero, and vanishes, together with all its derivatives, at some point of $[a,b]$. He first establishes some simple properties of singular functions; for example, a function is singular if and only if it has an infinite set of zeros (a zero of infinite order counting as an infinite set) but is not identically zero. He then rediscovers the theorem of S. Bernstein, that a function all of whose derivatives are positive in an interval is analytic, essentially by Bernstein's original method [Math. Ann. 75, 449–468 (1914); for references to more recent results, cf. Pólya, Bull. Amer. Math. Soc. 49, 178–191 (1943); these Rev. 4, 192]. He shows that a function having an infinite sequence of derivatives, each of which is of constant sign, cannot be singular. Let the class $C\{A_n\}$ be the class of functions of C^∞ such that $\lim\sup_{n\to\infty}\{A_n^{-1}\max|f^{(n)}(x)|\}^{1/n}<\infty$. He shows that, if $C\{A_n\}$ is not quasianalytic, then it contains a function vanishing, together with all its derivatives, precisely at the points of a prescribed closed nowhere dense set. *R. P. Boas, Jr.* (Providence, R. I.).

Differential Equations

Sispánov, Sergio. Differential equations analogous to those of Clairaut. Publ. Inst. Mat. Univ. Nac. Litoral 5, 127–152 (1945). (Spanish) [MF 13724]

Pompeiu, D. De la constante arbitraire dans les opérations d'intégration. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 235–241 (1941). [MF 13557]

Remarks of a formal character on the integration of some elementary differential equations. *R. P. Boas, Jr.*

Bouligand, Georges. Sur la méthode des approximations successives. Revue Sci. (Rev. Rose Illus.) 79, 605–607 (1941). [MF 13819]

An account of the work of Picard on differential equations and of Lebesgue on the Dirichlet problem, making connections between the methods of these authors and the modern theory of functional equations. *R. P. Boas, Jr.*

Barroso, Vergílio Simões. On the uniqueness of the solution of a system of ordinary differential equations in the classical case (and in the real domain). Gaz. Mat., Lisboa 6, no. 25, 6–8 (1945). (Portuguese) [MF 13324]

The system is of the classical form

$$y_i'=f_i(x;y_1,\cdots,y_m),\qquad i=1,\cdots,m,$$

where the f_i satisfy the usual hypotheses, including a Lipschitz condition. The proof of uniqueness is essentially that given in modern treatises, avoiding the use of the Picard approximations to the solution [see, for example, Bieberbach, Theorie der Differentialgleichungen, 3d ed., Springer, Berlin, 1932, pp. 31–32].

R. P. Boas, Jr. (Providence, R. I.).

Siegel, Carl Ludwig. Note on differential equations on the torus. Ann. of Math. (2) 46, 423–428 (1945). [MF 13429]

Without the use of the concept of rotation number the author gives a simplified proof of Denjoy's theorem on the topological mapping of a circle onto itself, $\theta\to\theta_1$ where $\theta_1=\phi(\theta)$. Denjoy's theorem states that, if $\phi'(\theta)$ is of bounded variation and not equal to zero, then the transformation is either periodic or ergodic. The author then generalizes considerably the application of this theorem to a differential equation on a torus. He proves that the system

$$dx/dt=f(x,y),\quad dy/dt=g(x,y),$$

where f and g are single-valued functions on the torus without common zero and with continuous derivatives of the second order, either has a periodic solution or else every solution is ergodic. *N. Levinson* (Cambridge, Mass.).

Wintner, Aurel. Small perturbations. Amer. J. Math. 67, 417–430 (1945). [MF 12922]

The author considers the linear system of differential equations

$$\dot{x}_i=\sum_{j=1}^{n}f_{ij}(t)x_j,\qquad i=1,\cdots,n.$$

Let $g_{ij}=|f_{ij}(t)-a_{ij}|$, where the a_{ij} are constants. The simplified system $\dot{y}_i=\sum a_{ij}y_j$ is assumed to have purely sinusoidal solutions. If all the g_{ij} satisfy (*) $\int^\infty g_{ij}dt<\infty$, then a solution $x_i(t)$ tends to some solution of the simplified system $y_i(t)$ as $t\to\infty$. The author's proof of this is very simple. Under weaker conditions than (*) such as $g(t)=O(t^{-k})$, $t\to\infty$, $1>k>0$, it is shown that $x_i(t)=O\{\exp(ct^{1-k})\}$. In particular, if $k=1$, $x_i(t)=O(t^c)$; in each case c is some positive constant. *N. Levinson* (Cambridge, Mass.).

Sointzev, G. On the asymptotic behaviour of integral curves of a system of differential equations. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 233–240 (1945). (Russian. English summary) [MF 13329]

Bendixon's results on the asymptotic behavior of an integral curve L of the system of differential equations $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ apply only when there is at most a finite number of singular points among the ω- or α-limit points of the curve. The author considers the general case. He obtains information about the structure of the sets of limit points of L, $E_\omega(L)$ and $E_\alpha(L)$. He also succeeds in generalizing Bendixon's theorem on the winding of an integral curve around a polycycle. *N. Levinson.*

Andronow, A., et Bautin, N. Sur un cas dégénéré du problème général de régulation directe. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 277–279 (1945). [MF 12948]

In certain direct control systems one is led to the differential equations $\dot{x}=f(x, y, z)$, $\dot{y}=z$, $\dot{z}=-\alpha-z$, $\alpha>0$, where $f=0$ between the planes $x-y=\pm\frac{1}{2}$ and $f\equiv z$ in the half-planes $x-y=\frac{1}{2}$, $z<0$ and $x-y=-\frac{1}{2}$, $z>0$. Considering this system in three-dimensional phase space, the authors show it can be reduced to the study of a transformation of a line into itself. They show that if $\alpha<3.04$ the system is stable, while if $\alpha>3.04$ it is self-excited and oscillates.

N. Levinson (Cambridge, Mass.).

Theodorchik, K. Theory of synchronization of relaxation autooscillatory systems. Acad. Sci. USSR. J. Phys. **9**, 139–143 (1945). [MF 13373]

The author elaborates an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 54–57 (1943); these Rev. **6**, 66]. The relationship of his work with that of König and Lamerey is indicated. *N. Levinson* (Cambridge, Mass.).

Teodorčik, K. Energy considerations for self-excited systems of Thompson's type with two degrees of freedom. Uchenye Zapiski Moskov. Gos. Univ. Fizika **74**, 67–72 (1944). (Russian) [MF 12719]

The author considers two circuits, each involving a tuned grid circuit G inductively coupled with another tuned circuit T. In one case the plate output is inductively coupled with G and in another case with T. In the first case the author is led to the pair of equations

$$\ddot{x}-2(\alpha-\beta x^2)\dot{x}+r_1^2x+s_1\dot{y}=0, \quad \ddot{y}+2\gamma\dot{y}+r_2^2y+s_2\dot{x}=0.$$

Assuming $x=a\sin wt$ and $y=b\sin wt-e\cos wt$, where a, b and e are slowly oscillating functions, an approximate system of equations is obtained for $\dot{a}$, $\dot{b}$, and $\dot{e}$ as well as an algebraic equation involving a, b, e and w. The stability of periodic solutions can be studied from this system. The second case is treated similarly. *N. Levinson.*

Rellich, Franz. Die zulässigen Randbedingungen bei den singulären Eigenwertproblemen der mathematischen Physik. (Gewöhnliche Differentialgleichungen zweiter Ordnung.) Math. Z. **49**, 702–723 (1944). [MF 11978]

Investigating boundary problems of ordinary second order differential equations $-(pu')'+qu=\lambda ku$ for an interval bounded by one singular point, say $x=0$, H. Weyl [Math. Ann. **68**, 220–269 (1910)] has found that one has to select at $x=0$ one out of infinitely many boundary conditions if the singularity is of the "limit circle" type. While the application of all these conditions as they are developed by M. H. Stone [Linear Transformations in Hilbert Space, Amer. Math. Soc. Colloquium Publ., vol. 15, New York, 1932] becomes rather difficult in general, the author observes that suitable restrictions on the coefficients p, q, k, as satisfied in most problems of mathematical physics, allow the establishment of boundary conditions which are as easy to handle as the usual conditions in the regular case. These restrictions mean, essentially, that $x=0$ is a point of definiteness of the differential equation for every value of the parameter λ.

It is shown first that, under these assumptions, the characteristic exponents ρ_1, ρ_2 at $x=0$ and some initial coefficients in the x expansion of a suitably chosen fundamental system are independent of λ. This leads to the construction of a subspace $\mathfrak{D}$ of the Hilbert space with elements which have a behavior near $x=0$ similar to that of the functions of the fundamental system. The author can now prove that the operator $k^{-1}\{-(pu')'+qu\}=Au$ satisfies the condition $(Au, v)=(u, Av)$ in the subspace of all those functions u of $\mathfrak{D}$ which satisfy a regular boundary condition at the second endpoint of the interval, supposed to be regular, while at $x=0$ they satisfy $u_0\cos\alpha+u_1\sin\alpha=0$ if ρ_1, ρ_2 are real, $u_0e^{i\alpha}+u_1e^{-i\alpha}=0$ if ρ_1, ρ_2 are not real. Here α is an arbitrary real number and u_0, u_1 are constants appearing in the expression of any function u of $\mathfrak{D}$ in terms of the fundamental system. Now in applying a Green's function, it can be shown that the operator $Au-iu$ has a completely continuous reciprocal in $\mathfrak{D}$. Therefore the differential equation with the boundary conditions as established has countably infinitely many proper values which converge to ∞, and a complete orthonormal system of proper functions.

The paper concludes by applying the theory to the problem of the circular membrane with fixed boundary and to Schroedinger's wave equation of the Kepler problem.

E. Hellinger (Evanston, Ill.).

Fedoroff, G. F. Sur le problème de Fuchs. Rec. Math. [Mat. Sbornik] N.S. **11**(53), 97–120 (1942). (Russian. French summary) [MF 12825]

The problem of Fuchs for differential systems has been studied by L. Schlesinger [J. Reine Angew. Math. **129**, 287–294 (1905); **141**, 96–145 (1912)], R. Fuchs [Math. Ann. **63**, 301–321 (1907)] and J. A. Lappo-Danilevsky [Mémoires sur la Théorie des Systèmes des Équations Differentielles Linéaires, vol. 3, Leningrad, 1936]. Fuchs considered the form which a linear equation of second order of Fuchsian type with four essentially singular points must have in order that the coefficients of the integral substitution (which a certain fundamental system of integrals undergoes when the independent variable describes a path around the singular points at a finite distance) should have values independent of the coordinate t of one of the singular points, considered as a parameter; in addition to the four essentially singular points 0, 1, t, ∞ there is at least one nonessentially singular point, with coordinate $\lambda(t)$ satisfying a nonlinear differential equation of second order. The author solves the problem of constructing a system having an integral matrix whose group is independent of t. It is shown that the nonessentially singular points are obtainable by a passage from the system to a single equation. It is established that there exists a differential equation of second order for the coordinate of the nonessentially singular point; the connection between the differential substitution and this coordinate is studied. Particular solutions are obtained for the fundamental nónlinear systems of Lappo-Danilevsky.

W. J. Trjitzinsky (Urbana, Ill.).

Kryloff, B. L. La construction des matrices canoniques intégrales du système de Gauss. Bull. Soc. Phys.-Math. Kazan (3) **12**, 83–117 (1940). (Russian. French summary) [MF 13903]

The author constructs canonical integral matrices for the Gaussian system $dY/dx=Y\{U_1/(x-a_1)+U_2/(x-a_2)\}$. It is shown that in the vicinity of a regular singular point there exists in every case a canonical integral matrix $K_j(x)=Y_j^{(0)}(x)\cdot\bar{K}_j(x)$, where $\bar{K}_j(x)$ is an analytic element for the point a_j and $Y_j^{(0)}(x)$ is determined by the characteristic values and the number of elementary divisors of the differential substitution U_j. In earlier works [same Bull. (3) **9**, 13–30 (1937); **11**, 181–197 (1938)] the author studied the problem on the basis of metacanonical matrices, when

the characteristic values do not differ by a nonzero integer; in the present general situation such matrices are not always available. The following cases may occur. (I) If U_j has distinct characteristic values $\xi_j^{(k)}$ $(k=1, 2)$, with $\xi_j^{(1)}-\xi_j^{(2)}$ not an integer, then $Y_j^{(0)}(x)=(x-a_j)^{[\xi_j^{(1)}, \xi_j^{(2)}]}$ (this is a notation for a diagonal matrix). (II) If the characteristic values ξ_j are equal and there are two elementary divisors, then $Y_j^{(0)}(x)=(x-a_j)^{[\xi_j, \xi_j]}$. (III) If the characteristic values ξ_j are equal and there is just one elementary divisor, then $Y_j^{(0)}(x)=(x-a_j)^{\alpha}$, where

$$\alpha=\begin{Vmatrix}\xi_j & 0\\ 1/\eta_j & \xi_j\end{Vmatrix}, \qquad \eta_j=2\pi i \exp(2\pi i\xi_j).$$

(IV) If the difference between the characteristic values is a nonzero integer, then $Y_j^{(0)}(x)=(x-a_j)^{[\xi_j^{(1)}, \xi_j^{(2)}]}$ or

$$Y_j^{(0)}(x)=\begin{Vmatrix}(x-a_j)^{\xi_j^{(1)}} & 0\\ (x-a_j)^{\xi_j^{(1)}}\log(x-a_j)/\eta_j & (x-a_j)^{\xi_j^{(2)}}\end{Vmatrix}.$$

W. J. Trjitzinsky (Urbana, Ill.).

Kryloff, B. L. Bestimmung der Gruppe eines Gauss'schen Systems in den Fällen, dass die charakteristischen Zahlen der Differentialsubstitutionen ganzzahlige Differenzen bilden. Bull. Soc. Phys. Math. Kazan (3) **12**, 119–138 (1940). (Russian. German summary) [MF 13960]

The author presents substantial and complete developments in the problem of determining the monodromic group for a Gaussian system (G) [cf. the preceding review]. In earlier works [same Bull. (3) **9**, 13–30 (1937); **11**, 181–197 (1938)], he studied cases when the characteristic values of the differential substitution of (G) do not differ by a nonzero integer. The present paper covers the general situation; generating substitutions of the monodromic group are given in all cases. A table indicates the transition from the generating substitutions of the monodromic group of (G) from any one case to any other case. *W. J. Trjitzinsky.*

Ritt, J. F. On the manifolds of partial differential polynomials. Ann. of Math. (2) **46**, 102–112 (1945). [MF 11793]

Étant donné un polynôme différentiel aux dérivées partielles F, dont on désigne par p la plus élevée des dérivées presentes, dans un arrangement déterminé, et par S le séparant $(S=\partial F/\partial p)$, on considère l'idéal Σ engendré par F et l'idéal Σ_1 des formes que, multipliées par S, produisent des formes de Σ. L'idéal Σ est alors le produit (intersection) de Σ_1 et d'un nombre fini d'autres idéaux, chacun desquels n'est contenu dans aucun autre et se comporte comme Σ_1 par rapport à une forme différentielle convenable. On appelle solution générale de F la variété des solutions de Σ_1. *B. Levi* (Rosario).

Levi, Howard. The low power theorem for partial differential polynomials. Ann. of Math. (2) **46**, 113–119 (1945). [MF 11794]

En imitant un procédé de calcul introduit par Ritt dans le mémoire analysé ci-dessus et généralisant aux formes aux dérivées partielles un raisonnement appliqué auparavant par l'auteur aux formes différentielles ordinaires [Trans. Amer. Math. Soc. **51**, 532–568 (1942); ces Rev. **3**, 264] on établit la condition nécessaire et suffisante pour que la solution générale d'un polynôme différentiel aux dérivées partielles soit une variété essentielle dans la variété d'un autre polynôme différentiel donné. *B. Levi* (Rosario).

Ritt, J. F. Analytical theory of singular solutions of partial differential equations of the first order. Ann. of Math. (2) **46**, 120–143 (1945). [MF 11795]

Les solutions singulières d'une équation aux dérivées partielles du premier order $F(x, y, z, p, q)=0$ peuvent être essentielles ou bien appartenir à la solution générale. Les théorèmes des deux mémoires analysés ci-dessus permettent d'affirmer que le premier cas se vérifie seulement quand la solution singulière considérée est définie par une équation $A(x, y, z)=0$ et de reconnaître s'il se vérifie effectivement. Dans ce cas l'auteur démontre que la solution est une enveloppe des solutions contenues dans la solution générale, généralisant ainsi un résultat établi depuis longtemps par Darboux dans un cas particulier. Dans le cas contraire la solution peut être approximée par des solutions non singulières. L'auteur généralise aussi la notion de caractéristique de manière que la solution singulière résulte remplie par ces courbes; il donne enfin quelques renseignements sur la généralisation au cas de plusieurs variables. *B. Levi.*

Gillis, P. Sur les équations linéaires aux différentielles totales. Bull. Soc. Roy. Sci. Liége **9**, 197–212 (1940). [MF 13046]

This paper is concerned with weakening the usual restrictive conditions under which it can be asserted that the total differential equation $dz=A(x, y, z)dx+B(x, y, z)dy$ has a unique solution which assumes a given value at a given point. The author employs a generalization of the classical condition for integrability, $A_y+A_zB=B_x+B_zA$, the generalized condition being expressed as a relation between line and surface integrals. [The exact meaning of the relation is not entirely clear to the reviewer.] A method of successive approximations is used to show that, if the generalized condition for integrability is satisfied in a suitable domain, and if A and B are of class C^0 in (x, y) and of class C^1 in z, then the equation has a unique solution. It is also shown that the equation has a unique solution if the generalized condition for integrability is satisfied and A and B are bounded and measurable and satisfy Lipschitz conditions with respect to z. In this case a solution of the equation is understood to mean an absolutely continuous function which satisfies the equations $z_x=A(x, y, z)$, $z_y=B(x, y, z)$ almost everywhere in the domain. These results are then extended to the case of a system of equations with m dependent and n independent variables. The paper concludes with some remarks concerning the relations of the preceding results to the problem of solving a Jacobian or complete system of partial differential equations. *L. A. MacColl.*

Gillis, Paul. Sur un théorème relatif aux formes différentielles intégrables. Bull. Soc. Roy. Sci. Liége **10**, 234–246 (1941). [MF 13063]

Let $\omega=udydz+vdzdx+wdxdy$. The differential form ω is said to be integrable in a three-dimensional domain D if $\int_\gamma\omega=0$ for every regular closed surface γ of D. If ω is integrable and u, v, w are continuous, it is known that there exist functions U, V, W such that

$$U+V+W=0, \quad U_{yz}=u, \quad V_{zx}=v, \quad W_{xy}=w$$

in D. The author shows that a similar result holds for an integrable form $\omega=A^{ij}dx_idx_j$ $(i, j=1, \cdots, 4; i\neq j)$ in a four-dimensional region. That is, if the $A^{ij}(x_1, \cdots, x_4)$ are continuous, there exist six functions U^{ij} such that

$$U^{12}+U^{23}+U^{31}=0, \quad U^{12}+U^{24}+U^{41}=0,$$
$$U^{13}+U^{34}+U^{41}=0, \quad U^{23}+U^{34}+U^{42}=0;$$
$$U^{12}_{x_1x_2}=A^{12}, \cdots, U^{34}_{x_3x_4}=A^{34}.$$

The results are extended to forms of degree p in n variables ($p \leqq n-1$). *F. G. Dressel* (Durham, N. C.).

Gillis, P. Sur les formes différentielles alternées. Bull. Soc. Roy. Sci. Liége 11, 464–470 (1942). [MF 13115]

A symbolic differential form of degree p

$$\omega_p = \sum a_{\alpha_1 \cdots \alpha_p}(x_1, \cdots, x_n)dx_{\alpha_1} \cdots dx_{\alpha_p}, \quad dx^i dx^j = -dx^j dx^i, \tag{1}$$

is called a C^k form if its coefficients are of class C^k. By generalizing the classical definitions of total, differentiable, and integrable forms, the paper states that certain theorems proved for C^1 and C^2 differential forms also hold for C^0 forms.

Let h^k be a k-dimensional regular multiplicity and fh^k its boundary. The form (1) is total in space D if there exists a form ω_{p-1} such that

$$\int_{fh^p} \omega_{p-1} = \int_{h^p} \omega_p$$

for all h^p in D. The form (1) is differentiable in space D if there exists a form ω_{p+1} such that

$$\int_{fh^{p+1}} \omega_p = \int_{h^{p+1}} \omega_{p+1}$$

for all h^{p+1} in D. In this case we write $\partial\omega_p = \omega_{p+1}$. Finally, a form ω_p is integrable in D if

$$\int_{fh^{p+1}} \omega_p = 0$$

for all h^{p+1} in D. With these definitions, in a Euclidean space D all integrable C^0 forms are total and, if ω_p and ω_q are two differentiable C^0 forms, then $\omega_p\omega_q$ is differentiable and

$$\partial(\omega_p\omega_q) = (\partial\omega_p)\omega_q + (-1)^p\omega_p(\partial\omega_q).$$

Let V be a closed variety. The following theorems of G. de Rham hold for integrable C^0 forms. (a) There always exists a C^2 form ω_p integrable on V having preassigned fundamental periods. (b) An integrable C^2 form ω_p is total if all its periods are zero. Applications are made to the theories of invariant integrals and the calculus of variations. Proofs are to appear elsewhere. *F. G. Dressel.*

Gillis, P. Sur quelques théorèmes de la théorie des formes différentielles alternées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 175–186 (1943). [MF 13842]

The author recalls certain of his earlier theorems on symbolic differential forms and makes general comments on them. [Cf. the two preceding reviews.]

F. G. Dressel (Durham, N. C.).

Lepage, Th.-H.-J. Quelques remarques sur les formes alternées intégrables. Bull. Soc. Roy. Sci. Liége 11, 510–518 (1942). [MF 13119]

After proving again some known results concerning symbolic forms

$$a_{i_1 \cdots i_p}dx^{i_1} \cdots dx^{i_p}, \quad dx^i dx^j = -dx^j dx^i,$$

the following theorems are proved for forms of the type

$$\Omega = f(x_1, \cdots, x_n; z, p_1, \cdots, p_n)dx^1 \cdots dx^n + f_{p_i}dx^1 \cdots dx^{i-1}\omega dx^{i+1} \cdots dx^n,$$

where $\omega = dz - p_i dx_i$, $f_{p_i} = \partial f/\partial p_i$. If ρ is the rank of the matrix $(f_{p_ip_j})$, then the rank of the form $d\Omega$ is $n+1$ if $\rho = 1$ and $n+\rho+1$ if $1 < \rho \leqq n$. Also, if $\rho = 1$, $d\Omega$ has $n+1$ distinct linear divisors; for $\rho > 1$, $d\Omega$ has $n-\rho+1$ distinct linear divisors.

The following lemma is also proved. If H is a symbolic form of degree and rank n, in $2n+1$ variables x_i, z, p_i, and if

$$H\omega \neq 0, \quad H\omega d\omega = 0, \quad (dH)\omega = 0, \quad \omega = z - p_i dx^i,$$

the associated system of $H\omega$ is completely integrable.

F. G. Dressel (Durham, N. C.).

Germay, R.-H. Étude, par la méthode d'approximations successives de M. L. Bruwier, des fonctions de Riemann associées à un système complètement intégrable d'équations aux différentielles totales de forme linéaire. Bull. Soc. Roy. Sci. Liége 11, 151–163 (1942). [MF 13097]

Consider the problem of finding a solution of the completely integrable system of total differential equations of the form

$$dz_j = \sum_{h=1}^{n}\left[\sum_{i=1}^{p} A_{ji}^{(h)}(x_1, \cdots, x_n)z_i + A_{j0}^{(h)}(x_1, \cdots, x_n)\right]dx_h, \quad j = 1, \cdots, p,$$

with initial condition $z_j(x_1^0, \cdots, x_n^0) = z_j^0$. The author gives the solution in terms of functions $G_i^{(\alpha,\beta)}$ having properties similar to those of Riemann's function for a hyperbolic differential equation. These functions are shown to satisfy a system of first order partial differential equations. The problem was treated previously by the author [Bull. Sci. Math. (2) 63, 272–278 (1939); these Rev. 1, 119]. The path of integration between the points $(x_1^0, \cdots, x_n^0)$ and $(x_1, \cdots, x_n)$ is not the same in the two papers.

F. G. Dressel (Durham, N. C.).

Lahaye, Edmond. Compléments à la note "Sur l'application de la méthode des approximations successives à la résolution des équations aux dérivées partielles linéaires du second ordre." Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 826–844 (1942). [MF 13681]

[The paper referred to in the title appeared in the same Bull. (5) 27, 537–551 (1941); these Rev. 4, 44.] Cauchy's problem is solved for second order partial differential equations that can be written in the form

$$\left\{A_0(x) + \sum_{i=1}^{n} A_i(x)\frac{\partial}{\partial x_i}\right\}\left\{\sum_{i=1}^{m} B_i(x)\frac{\partial u}{\partial x_i}\right\} = E(x), \ (x) \equiv (x_1, \cdots, x_n).$$

By using solutions of equations of the above types and successive approximations, Cauchy's problem is solved for the equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u + e(x) = 0.$$

F. G. Dressel (Durham, N. C.).

Kasner, Edward, and DeCicco, John. The Laplace equation in space. Proc. Nat. Acad. Sci. U. S. A. 31, 247–249 (1945). [MF 13290]

Kasner, Edward, and DeCicco, John. The Laplace equation. Science (N.S.) 102, 256–257 (1945). [MF 13295]

The ∞^1 surfaces $h(x, y, z) = C$, where $h(x, y, z)$ is harmonic, are said to form an isothermal family. In the first note the authors list some results of which the following are typical. (1) The system

$$\partial z/\partial x = p(x, y, z), \quad \partial z/\partial y = q(x, y, z)$$

possesses as solution an isothermal family of surfaces if and only if p and q satisfy the set of three partial differential

equations of second order,

$$p_y+qp_x=q_x+pq_x,$$

$$\tfrac{1}{2}(1+p^2+q^2)(p_{xx}+p_{yy}+p_{zz})=(p_yQ_x-p_xQ_y)+p(p_xP_x+p_yP_y+p_zP_z)+q(p_xQ_x+p_yQ_y+p_zQ_z),$$

$$\tfrac{1}{2}(1+p^2+q^2)(q_{xx}+q_{yy}+q_{zz})=(p_xQ_x-q_xp_x)+q(q_xP_x+q_yP_y+q_zP_z-p_xQ_x)+p(p_xQ_x+p_yQ_y+p_zQ_z).$$

(2) The only transformations of surface elements which send every isothermal field into an isothermal field are those of the Liouville inversive group. No proofs are given. In the second note the authors discuss briefly some of the results contained in the first. *M. O. Reade.*

Ridder, J. Über harmonische Funktionen. Mathematica, Timişoara **21**, 5–9 (1945). [MF 13969]

The author sketches a proof of the following theorem. The function $u(x, y)$ defined over a bounded region B of the (x, y)-plane is harmonic in B if, for every closed rectangle J lying in G and having sides parallel to the coordinate axes, $u(x, y)$ has finite partial derivatives in J which are summable over J and are linearly summable in x and y separately, and $\int(u_y dx-u_x dy)=0$ over the boundary of J. *L. H. Loomis* (Cambridge, Mass.).

Delange, H. Sur l'unicité de la distribution de masses produisant un potentiel donné. Bull. Sci. Math. (2) **69**, 7–12 (1945). [MF 13624]

The author gives a short proof of the known fact that, if we are given throughout the plane the logarithmic potential due to a mass distribution, then that distribution is uniquely determined; he also notes that the facts and the proof are similar in space. He cites earlier proofs by F. Riesz [Acta Math. **54**, 321–360 (1930)] and by C. de la Vallée Poussin [Les Nouvelles Méthodes de la Théorie du Potentiel, Actual. Sci. Ind. no. 578, Hermann, Paris, 1937, pp. 18–19]. *F. W. Perkins* (Hanover, N. H.).

Ciorănescu, Nicolas. Sur un problème pour les fonctions harmoniques dans un cercle. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **13**, 26–30 (1942). [MF 13563]

This paper contains a treatment of a specific problem of a general type previously considered by the author [Ann. Mat. Pura Appl. (4) **11**, 135–154 (1932)]. Let C_r and C_R be two concentric circles of radii r and R $(r<R)$, λ a real constant and $f(\theta)$ a real function. The author does not state fully his restrictions on $f(\theta)$; it is sufficient that $f(\theta)$ be periodic with period 2π and have a uniformly convergent Fourier series on the interval $0\leqq\theta\leqq 2\pi$. Consider the problem of determining a function $u(\rho, \theta)$ harmonic in C_R such that $u(R, \theta)-\lambda u(r, \theta)=f(\theta)$. If $\lambda\neq(R/r)^n$, $n=0, 1, 2, \cdots$, then there is a unique solution

$$u(\rho,\theta)=\frac{1}{\pi}\int_0^{2\pi}\left\{\frac{1}{2(1-\lambda)}+\sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n\frac{\cos n(\alpha-\theta)}{1-\lambda(r/R)^n}\right\}f(\alpha)d\alpha.$$

The author also expresses u as a power series in λ:

$$u(\rho,\theta)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\lambda^n\int_0^{2\pi}\frac{R^{2n+2}-r^{2n}\rho^2}{R^{2n+2}-2R^{n+1}r^n\rho\cos(\theta-\alpha)+r^{2n}\rho^2}f(\alpha)d\alpha.$$

Here the question of convergence is not fully investigated. A further study of the problem shows that, if a_n or b_n is the first Fourier constant of $f(\theta)$ which is not zero, then the power series converges when $|\lambda|<(R/r)^n$ but diverges when $|\lambda|\geqq(R/r)^n$. The author gives a third formulation of the solution. Under the same restrictions as in the second theorem, let $u_0(\rho, \theta)$ be the function harmonic in C_R determined by the boundary values $f(\theta)$ on C_R. Then $u(\rho, \theta)$ is the harmonic function determined by the boundary values

$$\phi(\theta)=\sum_{n=0}^{\infty}\lambda^n u_0((r/R)^n R, \theta)$$

on C_R.

The author also comments on the case in which $f(\theta)\equiv 0$. A necessary and sufficient condition that there exist a solution in this case other than the trivial solution $u\equiv 0$ is that λ be of the form $(R/r)^n$, $n=0, 1, 2, \cdots$. If $\lambda=(R/r)^n$ then any homogeneous harmonic polynomial of degree n is a solution, and these are the only nontrivial solutions.

Some of the formulas in the paper contain slight inaccuracies; two of these are corrected in this review. *F. W. Perkins* (Hanover, N. H.).

Stožek, Wladimir. Deux théorèmes sur le potentiel de simple couche. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 121–124 (1945). (French. Russian summary) [MF 13005]

The author discusses the following two theorems. (1) A necessary and sufficient condition that the logarithmic potential due to a simple distribution on the boundary of an ellipse be constant in the interior of the ellipse is that the linear density be proportional to the cube root of the curvature of the ellipse. (2) A necessary and sufficient condition that the Newtonian potential of a simple distribution on the surface of an ellipsoid be constant in the interior of the ellipsoid is that the superficial density be proportional to the fourth root of the Gaussian curvature of the ellipsoid. The author refers to related material in a memoir by J. Plemelj [Potentialtheoretische Untersuchungen, Preisschr. Fürstlich Jablonowskisch. Ges. **40**, no. 16, Teubner, Leipzig, 1911]. *F. W. Perkins* (Hanover, N. H.).

Uspensky, J. V. Sur la méthode de Laplace dans la théorie de l'attraction des ellipsoides homogènes. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 63–71 (1945) [MF 13722]

A simplified form of Laplace's proof of the theorem of Maclaurin is given. The theorem is that the attractions of two homogeneous confocal ellipsoids on an exterior point have the same direction and are proportional to their masses. *P. Franklin* (Cambridge, Mass.).

Fortet, Robert. Les progrès récents de la théorie du potentiel et de ses applications à l'analyse et la théorie des fonctions sous-harmoniques. Revue Sci. (Rev. Rose Illus.) **80**, 137–139 (1942). [MF 13818]

An expository article.

Vasilesco, Florin. Sur la notion de capacité d'un ensemble borné quelconque. Bull. Sci. Math. (2) **67**, 49–68 (1943). [MF 12630]

The author has generalized the notion of capacity and of a capacity distribution. If E is a bounded set and μ a positive distribution whose potential v is bounded on the closure of E, then the v-capacity is the least upper bound of the mass which can be distributed on E and whose potential nowhere exceeds v. A distribution on E whose mass is equal to the v-capacity of E is called a v-capacity distribution. The present paper is devoted to the problems arising if E is not closed. It shows the existence and uniqueness of a v-capacity distribution and studies the connection with the operation called "balayage." The author uses

sequences of closed subsets of E which approximate E in a definite way. *František Wolf* (Berkeley, Calif.).

Beckenbach, E. F. On almost subharmonic functions. Univ. Nac. Tucumán. Revista A. **4**, 243–254 (1944). [MF 13027]

[In the original, the author's name was misprinted B. E. Beckenbach.]

L'auteur démontre que, pour qu'une fonction localement sommable dans un domaine plan soit presque sousharmonique, il faut et suffit que la médiante spatiale (obtenue par moyenne dans l'aire d'un domaine circulaire δ_ρ^M de centre M et rayon ρ) soit sousharmonique quel que soit ρ, ou encore qu'elle soit, pour chaque M, majorée par la médiante périphérique (relative à la frontière de δ_ρ^M) pour presque tous les ρ dont le δ_ρ^M est contenu dans D. Propositions analogues pour les fonctions presque de classe PL. *M. Brelot* (Grenoble).

Tolotti, Carlo. Sul calcolo delle vibrazioni trasversali di un'asta elastica soggetta a sforzo assiale. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 13 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. **134**. [MF 11946]

Appelant respectivement $\lambda^{(\nu)}$, $\varphi^{(\nu)}$ les autovaleurs et les autofonctions de l'équation différentielle (1) $L[u]-\lambda\mu u=0$ avec

$$L[u]=\frac{d^2}{dx^2}\left\{EI\frac{d^2u}{dx^2}\right\}-\frac{d}{dx}\left\{\Pi\frac{du}{dx}\right\}+c\mu u$$

et avec les conditions aux limites $u=du/dx=0$ pour $x=0, 1$ ($EI(x)$, $\Pi(x)$, $\mu(x)$ fonctions données de x), toute solution de l'équation

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2u}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(\Pi\frac{\partial u}{\partial x}\right)+\mu\frac{\partial^2u}{\partial t^2}=0$$

avec

$$u=\partial u/\partial x=0,\quad x=0, 1,\quad [u]_{x=t}=f_0(x),\quad [du/dt]_{x=t}=f_1(x),$$

peut se représenter sous la forme $u(x,t)=\sum_{\nu=1}^{\infty}u^{(\nu)}(t)\varphi^{(\nu)}(x)$ dans le sens que le second membre converge au premier en moyenne. On donne des conditions suffisantes, relatives aux fonctions f_0, f_1, EI, Π, μ, pour que la convergence soit uniforme. On considère en particulier le cas où les fonctions EI, Π, μ se réduisent à des constantes et on démontre par comparaison que le spectre de (1) est borné inférieurement. *B. Levi* (Rosario).

Integral Equations

Rey Pastor, Julio. Partial integrals of functions of two variables in an infinite interval. Union Mat. Argentina. Memorias y Monografias (2) **1**, no. 3, 27 pp. (1944). (Spanish. French summary) [MF 13335]

The system

$$\int_{-\infty}^{\infty}f(x,y)dx=p(y),\quad \int_{-\infty}^{\infty}f(x,y)dy=q(x),$$

where p and q are given and f is to be found, has been studied by B. Levi [Publ. Inst. Mat. Univ. Nac. Litoral **1**, no. 1 (1939); these Rev. **2**, 99]. The author shows the existence of solutions belonging to various classes when p and q belong to appropriate corresponding classes, for example, the classes of polynomials, functions Lebesgue or Riemann integrable, or functions belonging to a specified Baire class. *R. P. Boas, Jr.* (Providence, R. I.).

Parodi, Maurice. Sur les propriétés de deux équations intégrales. C. R. Acad. Sci. Paris **220**, 76–77 (1945). [MF 13481]

It is known that the solution of the integral equation

$$\int_0^{\infty}J_0(2(yt)^{\frac{1}{2}})f_0(y)dy=g(t)$$

is

$$f_0(y)=\int_0^{\infty}J_0(2(sy)^{\frac{1}{2}})g(s)ds,$$

where $g(s)$ is the Laplace transform of $g(t)$. The author shows that the solution of the equation

$$t^{\nu/2}\int_0^{\infty}y^{-\nu/2}J_\nu(2(yt)^{\frac{1}{2}})f_\nu(y)dy=g(t),\quad \Re(\nu)>-\tfrac{1}{2},$$

is the ν-times repeated integral of $f_0(y)$. Likewise, the solution of

$$(\pi t)^{-\frac{1}{2}}\int_0^{\infty}e^{-\alpha^2/(4t)}x^{\nu/2}dx\int_0^{\infty}J_\nu(2(xy)^{\frac{1}{2}})F_\nu(y)y^{-\nu/2}dy=g(t)$$

is the ν-times repeated integral of $F_0(y)$. *M. Golomb.*

Hadwiger, H. Kleine Bemerkung zum Zinsfussproblem. Mitt. Verein. Schweiz. Versich.-Math. **45**, 31–35 (1945). [MF 13910]

Let $\Phi(x)\varepsilon L(0,\infty)$ be a given positive function. Put

$$\Phi(x)\phi(x,\alpha)=\int_0^{\infty}e^{-\alpha t}\Phi(x+t)dt.$$

The problem considered is that of expressing $\phi(x,\alpha)$ in terms of an initial value $\phi(x,\alpha_0)=\psi(x)$. The solution is given by

$$\phi(x,\alpha)=\psi(x)\int_0^{\infty}\exp\left\{-(\alpha-\alpha_0)t-\int_x^{x+t}d\xi/\psi(\xi)\right\}dt/\psi(x+t).$$

W. Feller (Ithaca, N. Y.).

Giraud, Georges. Équations de Fredholm dont le noyau est fonction holomorphe d'un paramètre. Bull. Sci. Math. (2) **65**, 103–112 (1941). [MF 13264]

Results stated previously [C. R. Acad. Sci. Paris **212**, 36–38 (1941); these Rev. **5**, 70] are proved. *F. G. Dressel* (Durham, N. C.).

Giraud, Georges. Opérations linéaires où figurent des intégrales principales simples et dont les données sont holomorphes par rapport à un paramètre. Bull. Sci. Math. (2) **65**, 144–155 (1941). [MF 12835]

The author proves that the results he obtained for principal value integral equations of the type

$$g(X)u(X)-\lambda\int_V G(X,A)u(A)ds_A=f(X)$$

[Ann. Sci. École Norm. Sup. (3) **56**, 119–172 (1939); these Rev. **1**, 145] also hold for principal value equations of the form

$$g(X,\lambda)u(X)-\int_V G(X,A;\lambda)u(A)ds_A=f(X),$$

where $g(X,\lambda)$ and $G(X,A;\lambda)$ are holomorphic functions of λ. *F. G. Dressel* (Durham, N. C.).

Schmeidler, Werner. Über Integralgleichungen erster Art mit beschränkter Kernmatrix. J. Reine Angew. Math. **186**, 16–24 (1944). [MF 12097]

The author considers integral equations of the form

$$(1)\qquad \int_a^b K(s,t)g(t)dt=f(s),$$

the restrictions on the kernel $K(s, t)$ being such that the entire discussion is applicable to equations of the form $Kg=f$, where f and g belong to two real Hilbert spaces and K is an arbitrary bounded transformation from one to the other. It is shown that, by appropriate choice of orthonormal systems (φ_α) and (ψ_β) in the two spaces, equation (1) can be reduced to a system of linear equations in an infinity of unknowns capable of explicit solution by recursion. The matrix of the system is not always an ordinary infinite matrix, but may contain an infinity of infinite submatrices.

It is shown that two transformations K_1 and K_2 are equivalent in the sense that they can be represented by the same matrix of the special form used earlier if and only if (a) the equations $K_1x=0$ and $K_2y=0$ have the same number of linearly independent solutions and (b) K_1K_1' and K_2K_2' (K' being the adjoint of K) are orthogonally equivalent, that is, there exists an orthogonal transformation U such that $UK_1K_1'U'=K_2K_2'$. An expansion theorem in terms of a generalisation of E. Schmidt's singular functions is proved. *F. Smithies* (Cambridge, England).

Pérès, Joseph. Fonctionnelles intervenant dans la discussion d'une équation de Fredholm. C. R. Acad. Sci. Paris **218**, 629–632 (1944). [MF 13471]

The author proposes to study integral equations of the form

$$\phi(M)-\int_S H(M, Q)\theta(Q)\phi(Q)dQ=f(M),$$

where $\theta(Q)$ replaces the constant parameter in the traditional Fredholm equations. He derives the relation

$$h_\theta(M, P)-h_\omega(M, P)=\int_S h_\theta(M, Q)\{\omega(Q)-\theta(Q)\}h_\omega(Q, P)dQ,$$

where $h_\theta(M, Q)\theta(Q)$ is the resolvent kernel of $H(M, Q)\theta(Q)$ (which exists except on a hypersurface in the space of the functions $\theta(Q)$, corresponding to the spectrum of eigenvalues in the Fredholm theory). This relation is analogous to the well-known reciprocity relation for the resolvent kernel of a Fredholm equation; a special case occurred in a recent paper on the theory of airfoils by L. Malavard [same C. R. **214**, 105–107 (1942); these Rev. **4**, 177].

The following result is also easily recognized by its Fredholm analogy. If the solution of the functional equation $f(M)=\mathfrak{F}[\phi, M]$ is

$$\phi(M)=-\int_S H(M, Q)f(Q)dQ,$$

then the solution of $f(M)=\mathfrak{F}[\phi, M]+\theta(M)\phi(M)$ is

$$\phi(M)=\int_S h_\theta(M, Q)f(Q)dQ.$$

M. Golomb (Philadelphia, Pa.).

Salechow, G. Eigenwertproblem für nichtlineare Integralgleichungen. Bull. Soc. Phys.-Math. Kazan (3) **12**, 175–181 (1940). (Russian. German summary) [MF 13907]

The author studies the existence problem for characteristic values of nonlinear integral equations of the Hammerstein type

$$\text{(H)}\qquad u(x)=\int K(x, y)f(y, u, \lambda)dy,$$

where $K(x, y)$ is symmetric and $\iint K^2(x, y)dxdy=c^2<\infty$. The background for this work is found, in part, in an investigation by V. Niemytzki [Rec. Math. [Mat. Sbornik] **41**, 421–438 (1934)]; the method is that of "fixed points." A value λ_1 is said to be a characteristic value of (H) if (H) has a solution, not identically zero, for $\lambda=\lambda_1$ and $f(y, 0, \lambda)\equiv 0$ for all λ. The class N consists of functions $f(y, u, \lambda)$ such that (1) $|f(y, u, \lambda)-\lambda u-s(y)|<\frac{1}{2}D/c$, where $s(y)\in L_2$, and (2) $|f(y, u_1, \lambda)-f(y, u_2, \lambda)|<\frac{1}{2}|u_1-u_2|/c$. Let $s(y)=0$. The following theorem is proved. For all λ which are characteristic values of the linear equation (L) $\psi(x)=\lambda\int K(x, y)\psi(y)dy$, (H) has a solution u (not identically zero) for all $f\in N$, such that $\|\psi-u\|_{L_2}<D$, where ψ is a solution of (L). *W. J. Trjitzinsky* (Urbana, Ill.).

Functional Analysis, Ergodic Theory

Barricelli, Nils Aall. Sur le prolongement à l'espace fonctionnel de la notion de volume et d'intégrale multiple. Arch. Math. Naturvid. **45**, no. 12, 131–154 (1942). [MF 12983]

The author applies three rules for passing from a finite dimensional space to a space of functions. In these a function of n variables becomes a functional, a sum relative to the coordinates becomes an integral and the index of the coordinates is permitted to assume a continuum of values. For instance, if a parallelepiped in function space is defined as the set of functions $x(t)$ for which $\varphi(t)\leqq x(t)\leqq\psi(t)$, then the volume according to these rules is

$$\exp\left\{\int_a^b \log(\psi(t)-\varphi(t))dt\right\}.$$

The rules are also applied to obtain integrals in special cases. Thus a polynomial function can be integrated in finite dimensions and the resulting formula "extended" to define an integral on the function space. [These rules are not new and the author does not show that the rules yield consistent results in various applications; nor does he seem aware of the fundamental difficulty with his definition of volume, that it corresponds in a finite dimensional space not to the volume of a parallelepiped, but to the nth root of the volume.] *F. J. Murray* (New York, N. Y.).

Germain, Paul. Définition des structures infinitésimales. C. R. Acad. Sci. Paris **220**, 344–345 (1945). [MF 13946]

Such a structure enables one to define differentiability for real-valued functions defined on a metric space.

R. Arens (Princeton, N. J.).

Gossieaux, Anne-Marie, et Papy, Georges. Un théorème sur les espaces du type *F*. Bull. Soc. Roy. Sci. Liége **13**, 146–150 (1944). [MF 13171]

This note proves for linear metric spaces a result proved by Schauder [Studia Math. **2**, 1–6 (1930)] for linear normed spaces. Let R be a dense subset of the sphere S of radius ρ and center a and let e be a point of S. Then there exist points r_i in R and real numbers c_i such that $\sum_{i<\infty} c_ir_i=e$ while $\sum_{i<\infty} c_i<1$. *M. M. Day* (Providence, R. I.).

Hyers, D. H., and Ulam, S. M. On approximate isometries. Bull. Amer. Math. Soc. **51**, 288–292 (1945). [MF 12270]

An ϵ-isometry of a space E with metric ρ into a space E' with metric ρ' is a transformation $T(x)$ such that $|\rho'(T(x), T(y))-\rho(x, y)|<\epsilon$. In the case where $E=E'$ and E is Euclidean or a Hilbert space, it is shown that there exists an isometry $U(x)$ such that $\|T(x)-U(x)\|<10\epsilon$. The

definition of ϵ-isometry is such that large spheres are changed less in shape than small ones; this is used to show the existence of $\lim_{n\to\infty} T(2^n x)/2^n$ for every x. This limit may then be taken as U. *F. J. Murray* (New York, N. Y.).

de Sz. Nagy, Béla. Sur les lattis linéaires de dimension finie. Comment. Math. Helv. **17**, 209–213 (1945).

Let R_n denote an n-dimensional real vector space and let P be a closed nondegenerate convex cone in R_n. The author shows that the following two properties which P might possess are equivalent. (a) There is a basis in R_n such that a vector is in P if and only if its components according to this basis are nonnegative. (b) Whenever u_1, u_2, v_1 and v_2 are in P and such that $u_1+u_2=v_1+v_2$ then there exist vectors $w_{11}, w_{12}, w_{21}, w_{22}$ in P such that $u_1=w_{11}+w_{12}$, $u_2=w_{21}+w_{22}$, $v_1=w_{11}+w_{21}$, and $v_2=w_{12}+w_{22}$. He then applies this result to obtain a new proof of the theorem of Youdine [C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 418–422 (1939)] to the effect that every finite dimensional linear lattice admits a basis such that a vector is nonnegative if and only if all of its components according to this basis are nonnegative. *G. W. Mackey* (Cambridge, Mass.).

McMillan, Brockway, and Lagerstrom, Paco. Extension of a theorem of Bochner on expressing functionals as Riemann integrals. Bull. Amer. Math. Soc. **51**, 251–258 (1945). [MF 12262]

In the theorem referred to [Ann. of Math. (2) **40**, 769–799 (1939); these Rev. **1**, 110] the assumption was made that the functional is defined for the constant function $f=1$. This is equivalent to the statement that the resulting Riemann measure is finite for the space itself. The present paper removes this restriction by a discussion which involves the actual construction of the measure function.

F. J. Murray (New York, N. Y.).

Dieudonné, Jean. Sur les homomorphismes d'espaces normés. Bull. Sci. Math. (2) **67**, 72–84 (1943). [MF 12631]

A linear transformation u between the vectorial spaces E and F is called a homeomorphism if the range of u is closed. If the manifold of zeros consists only of the zero element, u is called an isomorphism. Let E and F be complete. This paper is concerned with a "perturbation theory," that is, it deals with those properties of the transformation $u+w$ which occur if w is "sufficiently small." If w is linear with sufficiently small norm, $u+w$ is a homeomorphism with range F if u is. This also holds for the property of being an isomorphism with range F. The author also shows, by means of examples on polynomials, that these results do not necessarily hold when the spaces are not complete. Two other properties of u that carry over under perturbation are that of being a homeomorphism with a finite-dimensional zero manifold and that of being a homeomorphism whose range has a finite-dimensional complement. In the case of a not necessarily linear perturbation w, the author gives sufficient conditions that the range of $u+w$ contain a sphere. This result is applied in the linear case to show that $u+w$ is also a homeomorphism, that for each zero of $u+w$ there is a zero of u in a neighborhood whose size is specified and that a complement to the manifold of zeros of u will also be related in this way to $u+w$. Certain dual results to these are also given. *F. J. Murray.*

Dieudonné, Jean. Sur le théorème de Hahn-Banach. Revue Sci. (Rev. Rose Illus.) **79**, 642–643 (1941). [MF 13402]

The author uses the method of ordered groups to prove the following corollary of the Hahn-Banach theorem [S. Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 27, theorem 1]. Let K be a convex set not containing the origin in a linear space E, and assume that there is a point x_0 of K such that every line through x_0 meets K in a segment of which x_0 is not an end point; then there exists a distributive functional f defined in E such that $f(x)<0$ for every x in K. *M. M. Day* (Providence, R. I.).

Dieudonné, Jean. Sur la séparation des ensembles convexes dans un espace de Banach. Revue Sci. (Rev. Rose Illus.) **81**, 277–278 (1943). [MF 13401]

Tukey [Portugaliae Math. **3**, 95–102 (1942); these Rev. **4**, 13] proved that, if two bounded closed convex sets in a reflexive Banach space have no common points, then there exists a closed hyperplane separating the two sets. This note shows, by means of an example in l_1, that the hypothesis of reflexivity cannot be dropped completely.

M. M. Day (Providence, R. I.).

Fortet, R. Remarques sur les espaces uniformément convexes. Bull. Soc. Math. France **69**, 23–46 (1941). [MF 13238]

This paper contains a variety of results depending on the notion of normality in a uniformly convex Banach space. If a point x and a closed linear manifold V are given in a complex Banach space B, a normal projection of x on V is a point y of V such that $\|x-y\|$ is the distance from x to V. (1) If B is uniformly convex, the normal projection of x on V always exists and is unique. The point x is called normal to V if the normal projection of x on V is the origin in B; hence x is called normal to y if $\|x-\lambda y\|\geqq\|x\|$ for all λ. A uniformly convex B is said to have a regular norm (uniformly regular norm) if $(\|x-\lambda y\|-\|x\|)/\lambda\to 0$ as $\lambda\to 0$ whenever $\|x\|>0$ and x is normal to y (and the convergence is uniform for all $\|x\|=\|y\|=1$). (2) If B is uniformly convex and of regular norm, and if normality is a symmetric relationship between points of B, then the normal projection on any closed linear V is a (continuous) linear operation. (3) Each space l_p, $p>1$, has a uniformly regular norm (but normality is not symmetric). (4) If B is uniformly convex and of uniformly regular norm, the conjugate space B^* also has these properties. *M. M. Day.*

Murray, F. J. Quasi-complements and closed projections in reflexive Banach spaces. Trans. Amer. Math. Soc. **58**, 77–95 (1945). [MF 12773]

If $\mathfrak{M}$ and $\mathfrak{N}$ are additive subsets of a Banach space $\mathfrak{B}$, $\mathfrak{N}$ is said to be a quasi-complement of $\mathfrak{M}$ in case $\mathfrak{M}\mathfrak{N}=(0)$ and $\mathfrak{M}\cdot+\cdot\mathfrak{N}$ (that is, the set of vectors of the form $x+y$, $x\in\mathfrak{M}$, $y\in\mathfrak{N}$) is dense in $\mathfrak{B}$. It is shown that for a separable reflexive Banach space $\mathfrak{B}$ every closed additive set $\mathfrak{M}$ has a quasi-complement $\mathfrak{N}$. A quasi-complement $\mathfrak{N}$ of $\mathfrak{M}$ which is not a complement has a proper subset $\mathfrak{N}''$ which is also a quasi-complement of $\mathfrak{M}$, and $\mathfrak{N}$ is a proper subset of a set $\mathfrak{N}'$ which is a quasi-complement of $\mathfrak{M}$. Corresponding to quasi-complements we have closed additive projections with domain dense and to every closed additive subset $\mathfrak{M}$ of a separable reflexive Banach space corresponds a closed additive projection with domain dense, whose range is $\mathfrak{M}$.

N. Dunford (New Haven, Conn.).

Grinblum, M. M. Biorthogonal systems in Banach space. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 75–78 (1945). [MF 13742]

A complete bounded set $\{x_i\}$ in a separable Banach space E is called strictly minimal if no element is in the span of the others; a basis is an example. Such a sequence has associated with it a unique biorthogonal set of functionals F_i. If the latter are also uniformly bounded, the pairs $\{x_i, F_i\}$ form a regular biorthogonal system. Associated with regular sets are sequences $\{\epsilon_i\}$ such that, if $\{\eta_i\}$ is a sequence of elements of E with $\|\eta_i\| < \epsilon_i$, then $x_i + \eta_i = y_i$ is associated with a biorthogonal set G_i, with $\{y_i, G_i\}$ regular. The set M of all complete bounded strictly minimal sequences forms a topological space with an ϵ-neighborhood of $\{x_i\}$ defined by $\{x_i + \eta_i\}$, where $\sum_1^\infty \|\eta_i\| < \epsilon$. If $\{x_i\}$ is a basis in E, then every sequence in a sufficiently close neighborhood (in the preceding topology) is also a basis. No mention is made of the work of several other writers along these lines. [For references, cf. Pollard, Ann. of Math. (2) 45, 738–739 (1944); these Rev. 6, 127.] *H. Pollard.*

Van der Lijn, G. Quelques formules concernant les opérateurs polynomiaux. Bull. Soc. Roy. Sci. Liége 11, 528–531 (1942). [MF 13122]

The formulas considered are the binomial theorem for a monomial operator and the relationship between a polynomial operator and the corresponding multilinear operator. *F. J. Murray* (New York, N. Y.).

Fan, Ky. Sur le théorème d'existence des équations differentielles dans l'analyse générale. Bull. Sci. Math. (2) 65, 253–264 (1941). [MF 13258]

Let E_1 be the space of the real numbers, E_2 a Banach space, x and y points of E_1 and E_2, respectively. With each couple (x, y) of a neighborhood N of the couple (x_0, y_0) let there be associated a mapping which maps the vector v of E_1 into the vector $\Psi_{(x,y)}(v)$ of E_2; Ψ is linear in v. Let $\partial_v{}^x f(x)$ denote the Fréchet differential of the function $f(x)$ at the point x corresponding to the increment v of x. The author considers the equation $\partial_v{}^x f(x) = \Psi_{(x,f(x))}(v)$, which is a generalization of an ordinary differential equation, and proves the existence of a solution f with the initial condition $f(x_0) = y_0$ in a neighborhood $N' \subset N$ of (x_0, y_0). If the vector $\Psi_{(x,y)}(v)$ is denoted by $F(x, y)$, where v is the vector $\overline{01}$, the main assumptions are the following: (i) F is continuous in (x, y), (ii) F satisfies a Lipschitz condition with respect to y, (iii) the norm of F is bounded in N.

The proof consists of a generalization of Cauchy's polygon method which is classical for ordinary differential equations. A similar existence theorem was proved by M. Kerner [Prace Mat.-Fiz. 40, 47–67 (1932)] by a different method (successive approximations). However, Kerner used stronger hypotheses, for example, the existence of the partial derivative with respect to y instead of assumption (ii). A paper by Michal and Elconin [Acta Math. 68, 71–107 (1937)] deals with the more general case where E_1 is also an abstract space (generalization of partial differential equations). However, if specialized to the present case their hypotheses are again stronger. [It should be noted, however, that both Kerner and Michal and Elconin also prove the uniqueness of the solution.] The author concludes with an application to systems of countably many ordinary differential equations. *E. H. Rothe* (Ann Arbor, Mich.).

Lorch, Edgar R. The Cauchy-Schwarz inequality and self-adjoint spaces. Ann. of Math. (2) 46, 468–473 (1945). [MF 13431]

Let B be a Banach space, let B^* be its conjugate and suppose that there exists a topological and algebraic isomorphism T of B on B^*. The author shows that, if T has the additional property that for all f in B the equality $[T(f)](f) = \|f\| \|T(f)\|$ holds, then B is a Hilbert space (not necessarily separable or infinite dimensional) and that if T is suitably normalized then $\|f\| = \|T(f)\|$ and T is the mapping of the Hilbert space on its conjugate defined by its inner product. The proof in the finite dimensional case is based on a lemma concerning inequalities and the extension to the infinite dimensional case is immediate. Real and complex spaces are treated separately. *G. W. Mackey.*

Julia, Gaston. Décomposition en produit infini des opérateurs linéaires de l'espace hilbertien. J. Math. Pures Appl. (9) 20, 347–362 (1941). [MF 13365]

The author considers the problem of expressing a linear operator A, defined on a Hilbert space H, as an infinite product of canonical operators. He assumes that A is defined on a domain D_A everywhere dense in H and that ϵ_n is a complete orthonormal base in D_A. He then orthonormalizes by Schmidt's well-known method the sequence $A_n = Ae_n$ and obtains expressions of the form $A_n = \alpha_{n1}\epsilon_1 + \cdots + \alpha_{nn}\epsilon_n$, where the ϵ_i are the orthonormal system and $\alpha_{ij} = 0$ for $i < j$. Consider now the operators B and U defined by the relations $A_n = B\epsilon_n$, $\epsilon_n = Ue_n$. If ϵ_n is complete, U is unitary and $A = BU$. In case ϵ_n is not complete, U is such that $U^*U = 1$, $UU^* = P_v$, where P_v is the projection associated with the linear closed manifold $V = [\epsilon_1, \epsilon_2, \cdots]$. The relation $A = BU$ is still valid. The author now defines a sequence B_i of operators as follows: $B_1\epsilon_1 = A_1$, $B_1\epsilon_n = \epsilon_n$ $(n \geqq 2)$; $B_p\epsilon_p = A_p$, $B_p\epsilon_n = \epsilon_n$ $(n \neq p)$. The product $B_pB_{p-1}\cdots B_1$ is bounded, leaves invariant all ϵ_n with $n > p$ and is equal to $B^{(p)} + 1 - E_p$, where $B^{(p)}$ is the reduced operator of order p of B and E_p is the projection associated with $[\epsilon_1, \cdots, \epsilon_p]$. When A is bounded, $B_pB_{p-1}\cdots B_1$ converges strongly with p to B and hence $A = \prod B_jU$. He then shows that the spectrum of each B_p consists of the two points 1 and α_{pp} and proceeds to get an expression for the bound of B_p. He turns now to the case where A is not bounded and defines the domain of strong convergence d_B of B as all $\xi = \sum \xi_n\epsilon_n$ such that $\sum \xi_nA_n$ converges strongly. It follows that for each ξ in d_B the series $\sum \xi_nA_n$ and the product $B_p\cdots B_1\xi$ converge strongly to the same limit. Furthermore, it is shown that the product $B_p\cdots B_1UX$, the series $\sum_1^\infty x_nA_n$ and the expression $B^{(p)}Ux$ converge to $\sum x_nA_n$ at every point x in d_A.

H. H. Goldstine (Philadelphia, Pa.).

Cooper, J. L. B. The spectral analysis of self-adjoint operators. Quart. J. Math., Oxford Ser. 16, 31–48 (1945). [MF 12789]

The author objects to the self-adjointness condition for the existence of a spectral resolution, since it "does not convey any direct physical notion at all." He introduces the condition (a): for every φ in a dense subset of D, the domain of H, there exists a function $\psi(t)$ of a real variable t, $-\infty < t < \infty$, with values in the domain of H, for which $\psi(0) = \varphi$ and $-id\psi/dt = H\psi$. He shows that a symmetric H which satisfies (a) has a spectral resolution and that a self-adjoint transformation satisfies (a). It seems to the reviewer that what the author is doing is to replace the study of the set of unitary transformations $\exp(iH)$ by

analytic methods from Fourier analysis. The reviewer remains convinced that the geometric approach, which is the justification for abstract Hilbert space, would yield the "direct physical notions" in a much simpler way.

F. J. Murray (New York, N. Y.).

Hamburger, Hans Ludwig. On a class of Hermitian transformations containing self-adjoint differential operators. Proc. Nat. Acad. Sci. U. S. A. **31**, 185–189 (1945). [MF 12795]

Let $\mathfrak{L}_2(\Delta)$ be the space of all real and complex functions defined in a closed interval Δ of the x-axis such that $|f(x)|^2$ is Lebesgue integrable in Δ. The author defines a family of projections P_s in $\mathfrak{L}_2(\Delta)$, where s is an interior point of Δ, by $P_sf(x)=f(x)$ for $x\leqq s$, $P_sf(x)=0$ for $x>s$, $x\epsilon\Delta$. Let H be a closed Hermitian transformation with domain $\mathfrak{D}\subset\mathfrak{L}_2(\Delta)$ and let $\mathfrak{D}_s$ be the set of all $f(x)$ such that both $f(x)$ and $P_sf(x)$ are elements of $\mathfrak{D}$. If $\mathfrak{D}_s$ is everywhere dense in $\mathfrak{D}$, there exists a contraction ${}^sH$ of H with domain $\mathfrak{D}_s$. The author calls a closed Hermitian transformation of deficiency index (m, m) with domain $\mathfrak{D}$ an mth order transformation of class P if, (i) when $\{\varphi_s\}$ and $\{\psi_s\}$ are two sets of m linearly independent eigensolutions of H^* (the adjoint of H) corresponding to the eigenvalues i and $-i$, respectively, the two sets of m values $\{(P_t-P_s)\varphi_s\}$ and $\{(P_t-P_s)\psi_s\}$ are both linearly independent for every $s\neq t$ of Δ; (ii) for every s interior to Δ, $\mathfrak{D}_s$ is everywhere dense in $\mathfrak{D}$ and the contraction ${}^sH$ of H is of deficiency index $(2m, 2m)$; and (iii) ${}^sHP_sf=P_s{}^sHf$ for every f of $\mathfrak{D}_s$. In the main theorem the author gives the conditions that, when H is an mth order transformation of class P, H^* can be represented by a self-adjoint differentiable operator of mth order. In a later paper proofs will be given as well as methods of obtaining all transformations of class P. The author concludes by defining a class P of Hermitian transformations given in any abstract Hilbert space and obtaining the theorem corresponding to the main theorem mentioned above.

J. Williamson (Flushing, N. Y.).

Millsaps, Knox. A note on generalized Hilbert space. Univ. Nac. Tucumán. Revista A. **4**, 317–320 (1944). [MF 13030]

The author discusses, with only indications of proofs, the spaces of real sequences $\{x_n\}$ such that $\sum|x_n|^\alpha<\infty$, under a generalized norm $(\sum|x_n|^\alpha)^\beta$. Most of the results are concerned with the case $\beta=1/\alpha$, $\alpha\geqq 1$, and are known. [See, for example, Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 12.] *H. Pollard* (New Haven, Conn.).

Ambrose, Warren. Structure theorems for a special class of Banach algebras. Trans. Amer. Math. Soc. **57**, 364–386 (1945). [MF 12561]

The particular type of Banach algebra (normed ring) considered is one in which the underlying complex Banach space is a Hilbert space A. In addition there exists for every $x\epsilon A$ an $x^*\epsilon A$ satisfying $(xy, z)=(y, x^*z)$ for all y, z in A. Commutativity of A or the existence of a unit is not assumed. The author obtains the complete structure theorems for these H^* algebras. Some examples are: (1) The full matrix algebra with complex elements $a=a(i, j)$, where the norm is defined by $\|a\|^2=\sum_{i,j}|a(i, j)|^2$; here i, j vary over some set of indices. Thus the definition of the norm qualifies the "full." If the set of indices is infinite, the full algebra will not have a unit. (2) The algebra $L_2(G)$ where G is a compact topological group and L_2 is the Hilbert space over G; in this case, multiplication is defined in terms of convolution. Essentially, a proper H^* algebra is one without radical; more precisely it is an A in which $xA=(0)$ (or $Ax=(0)$) implies $x=0$. The fundamental structure theorem is that every proper H^* algebra is the direct sum of minimal two-sided ideals; each such ideal is a full matrix algebra. Thus example (1) is the prototype of all H^* algebras. Example (2) is a special H^* algebra in which the transformations of its regular representation are completely continuous. For such an algebra the minimal two-sided ideals have finite dimensionality. The development of the paper pivots about the investigation on idempotents. These are defined by $e^*=e$ as well as $e^2=e$ (thus they correspond to orthogonal projections). An e may be constructed as follows. If x is self-adjoint, and as a left multiplicative operator is nonnegative definite and has precisely the bound 1, then the sequence $\{x^n\}$ converges to an $e\neq 0$. Now easy examples show that this phenomenon is not duplicated in the general spectral theory of self-adjoint operators where uniform convergence does not obtain. It should perhaps be stressed that the feature which makes this possible in A is that in a ring the bound of the multiplicative operator x stands in a very special relation to the norm of x.

E. R. Lorch (New York, N. Y.).

Lewitan, B. Normed rings generated by the generalized operation of translation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 3–6 (1945). [MF 13748]

Let Z be a locally compact space which satisfies the second countability axiom and in which there exists a measure $m(\Delta)$ positive for open sets. Let L denote the B-space of absolutely integrable complex-valued functions on Z. The author assumes for each $s\epsilon Z$ the existence of certain bounded linear operators T^s and $\bar{T}^s$ on L which are generalizations of the translations $T_t^sf(t)=f(t+s)$ and $\bar{T}_t^sf(t)=f(t-s)$ for the case in which Z is an (additive) Abelian topological group and $m(\Delta)$ is the Haar measure. Multiplication can then be defined in L by the "convolution"

$$fg(t)=\int_Z \bar{T}_t^sf(t)g(s)dm(s)$$

so that L becomes a normed ring. Following Gelfand and Raikov [same C. R. (N.S.) **28**, 195–198 (1940); these Rev. **2**, 217], who considered the group case, the author studies the ring R obtained by adjoining a unit to L so that L is a maximal ideal in R. It is shown that there exists a one-to-one correspondence between the maximal ideals (other than L) in R and continuous characteristic functions of the operators T^s, that is, functions $\varphi(t)$ such that $T_t^s\varphi(t)=\varphi(s)\varphi(t)$.

C. E. Rickart (New Haven, Conn.).

Lewitan, B. The theorem on the representation of positively definite functions for the generalized operation of translation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 159–161 (1945). [MF 13735]

A bounded continuous function $p(t)$ on Z [see the preceding review for notations] is defined to be positive definite if for any points $t_1, \cdots, t_n$ in Z and any complex numbers $\rho_1, \cdots, \rho_n$ it is true that

$$\sum_{i,j=1}^{n}\bar{T}_{t_i}^{t_j}p(t)\rho_i\bar{\rho}_j\geqq 0.$$

Following Raikov [same C. R. (N.S.) **28**, 296–300 (1940); these Rev. **2**, 223], who considered the group case, the author proves that a continuous positive definite function

$p(t)$ is representable in the form

$$p(t)=\int_{\mathfrak{M}-L}\overline{\varphi(\alpha,t)}dF(\alpha),$$

where $\mathfrak{M}$ is the space of maximal ideals in R, $F(\Delta)$ is a completely additive function on Borel sets of the locally compact set $\mathfrak{M}-L$, and $\varphi(\alpha,t)$ is a continuous characteristic function of the operators T^t. This generalizes further the original theorem of Bochner which applies in the case where Z is the additive group of real numbers. The theorem is also applied to obtain a spectral representation for certain families A_t ($t\in Z$) of operators on Hilbert space.

C. E. Rickart (New Haven, Conn.).

Lewitan, B. Plancherel's theorem for the generalized translation operator. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 318–321 (1945). [MF 14016]

Using results developed in the two papers reviewed above, the author obtains the following generalization of Plancherel's theorem. A unique completely additive measure $F(\Delta)$ can be defined on $\mathfrak{M}-L$ such that the operators

$$Pf=\int_Z f(t)\overline{\varphi(\alpha,t)}dm(t),\quad Qg=\int_{\mathfrak{M}-L}g(\alpha)\varphi(\alpha,t)dF(\alpha)$$

are defined for all the square-integrable functions $f(t)$, $g(\alpha)$ (on the corresponding spaces) and are mutually inverse and isometric; that is, $PQ=QP=1$,

$$\int_{\mathfrak{M}-L}|Pf|^2dF(\alpha)=\int_Z|f(t)|^2dm(t),$$

$$\int_Z|Qg|^2dm(t)=\int_{\mathfrak{M}-L}|g(\alpha)|^2dF(\alpha).$$

The group case was considered by M. Krein [same C. R. (N.S.) **30**, 484–488 (1941); these Rev. **2**, 316].

C. E. Rickart (New Haven, Conn.).

Michal, Aristotle D. The total differential equation for the exponential function in non-commutative normed linear rings. Proc. Nat. Acad. Sci. U. S. A. **31**, 315–317 (1945). [MF 13439]

The author shows that in a ring of the indicated type, the function $\exp(A)=I+A+A^2/2!+\cdots$ is the unique entire analytic solution $z(A)$ of the following system:

$$\delta z(A)=\int_0^1 z((1-\xi)A)\cdot\delta A\cdot z(\xi A)d\xi$$

with $z(0)=I$, the unit of the ring. The functions are analytic in a sense defined by A. D. Michal and R. S. Martin [J. Math. Pures Appl. (9) **13**, 69–91 (1934)]. The symbol $\delta z(A)$ represents a Fréchet differential. The proof is based on properties established by the author for the "power series" which define these analytic functions.

E. R. Lorch (New York, N. Y.).

Cameron, R. H., and Martin, W. T. Transformations of Wiener integrals under a general class of linear transformations. Trans. Amer. Math. Soc. **58**, 184–219 (1945). [MF 13308]

Let C be the space of all real continuous functions $x(t)$ defined for $0\leqq t\leqq 1$ and vanishing when $t=0$. The authors use Wiener's definition of measure (Brownian movement measure) on C-sets. They consider the transformation

$$y(t)=x(t)+x_0(t)+\int_0^1 K(t,s)x(s)ds$$

which takes C into itself. Here $K(t,s)$ and $x_0(t)$ are fixed functions satisfying appropriate regularity conditions. An expression is found for the measure of the image of a measurable subset of C. As a simple application of this and related results, the characteristic function of the chance variable $\int_0^1 x(t)^2dt$ is evaluated.

J. L. Doob.

García, Mariano. Orbit-components and component orbits. Duke Math. J. **12**, 473–483 (1945). [MF 13516]

Let $f(X)=X$ be a homeomorphism of a separable metric space X. By the orbit-component of a point $x\in X$ is meant the set of points y such that every open invariant set containing y overlaps every open invariant set containing x. The invariant set swept out by any connected set in X is called a component orbit. The main purpose of the paper is to study relations between these notions and recurrence properties of f. Two typical results are: (1) In a compact space, the closures of the component orbits determined by the components of X give an upper semi-continuous decomposition of X if and only if f is componentwise recurrent on X. (2) If X is compact and f is pointwise recurrent on X, then every orbit-component in X is the closure of a component orbit.

J. C. Oxtoby (Bryn Mawr, Pa.).

Fréchet, Maurice. Les fonctions asymptotiquement presque-périodiques. Revue Sci. (Rev. Rose Illus.) **79**, 341–354 (1941). [MF 13822]

The principal results have been previously announced without proof [C. R. Acad. Sci. Paris **213**, 520–522 (1941); these Rev. **5**, 96]. In addition, the author develops the properties of asymptotically almost periodic vectors, which are defined as vectors depending on a parameter t, the components of the vector being asymptotically almost periodic functions. The results are closely analogous to those obtained for almost periodic functions in the sense of Bohr.

G. A. Hedlund (Charlottesville, Va.).

Fréchet, Maurice. Une application des fonctions asymptotiquement presque-périodiques à l'étude des familles de transformations ponctuelles et au problème ergodique. Revue Sci. (Rev. Rose Illus.) **79**, 407–417 (1941). [MF 13821]

This paper contains the proofs of a number of results which have been announced previously [C. R. Acad. Sci. Paris **213**, 607–609; these Rev. **5**, 189]. The properties of asymptotically almost periodic functions and vectors [see the preceding review] are applied to continuous flows (T_t, $-\infty<t<\infty$) and semiflows (T_t, $0\leqq t<\infty$) in a bounded closed set S in m-dimensional Euclidean space. The author obtains what he calls a generalization of Birkhoff's lemma by omitting hypotheses concerning measures, and in the case of a semiflow by omitting the condition that the inverse of T_t exists, but by adding the very strong hypothesis that T_tM is equicontinuous in M for all t involved. He shows that, as a consequence,

$$\text{(A)}\qquad \lim_{L\to\infty}L^{-1}\int_0^L f(T_tM)dt,$$

where f is continuous on S, exists for all M in S. If a measure is defined in S and metric transitivity is assumed, the mean (A) is shown to be the same for almost all points

of S and the time average is equal to a space average almost everywhere. In addition, some results concerning the Fourier analysis of $f(T_tM)$ are obtained and it is shown that in the case of a flow there exists an invariant measure. [These results are closely related to those obtained (independently, later and simply) by Hartman and Wintner [Amer. J. Math. 65, 273–278 (1943); these Rev. 4, 226].]
G. A. Hedlund (Charlottesville, Va.).

Fréchet, Maurice. Sur le problème ergodique. Revue Sci. (Rev. Rose Illus.) 81, 155–157 (1943). [MF 13805]
The author extends the results of a previous paper [see the preceding review] by observing that the hypothesis of metric transitivity is unnecessarily strong and that the time average equals a space average everywhere and not merely almost everywhere. A correction to a preceding paper is made. *G. A. Hedlund* (Charlottesville, Va.).

Theory of Probability

Fréchet, Maurice. Valeurs moyennes attachées à un triangle aléatoire. Revue Sci. (Rev. Rose Illus.) 81, 475–482 (1943). [MF 13802]
This paper contains a study of the mean values of certain functions of triangles when the triangles are chosen at random. One function is the ratio of the area to the square of the perimeter and another is the square of that ratio. Of course the mean values depend upon the particular mechanism by which the random selection is accomplished; the author considers three such mechanisms. He performs a number of experiments in each case and compares the observed results with the theoretical means.
A. H. Copeland (Ann Arbor, Mich.).

Frame, J. S. Mean deviation of the binomial distribution. Amer. Math. Monthly 52, 377–379 (1945). [MF 13382]
This paper contains a formula relating the mean deviation of a binomial distribution to the largest term of a related binomial expansion. Thus if the probabilities constituting the distribution are the terms of the expansion of $(q+p)^n$, then the mean deviation of the distribution is npq times the largest term of the expansion of $(q+p)^{n-1}$.
A. H. Copeland (Ann Arbor, Mich.).

von Schelling, H. Eine Formel für die Teilsummen gewisser hypergeometrischer Reihen und deren Bedeutung für die Wahrscheinlichkeitstheorie. Naturwissenschaften 30, 757–758 (1942). [MF 13685]
It is shown that the summatory function and the truncated moments of the Pólya distribution can be expressed in terms of partial sums of a hypergeometric series (that is, to say, by generalized binomial coefficients).
W. Feller (Ithaca, N. Y.).

Risser, René. Sur l'équation caractéristique des surfaces de probabilités. C. R. Acad. Sci. Paris 220, 31–32 (1945). [MF 13479]
A formal way is described in which Pearson curves can be connected with a classical urn-scheme with balls of three colors. *W. Feller* (Ithaca, N. Y.).

Risser, René. Sur le mode de tirages contagieux. C. R. Acad. Sci. Paris 220, 210–212 (1945). [MF 13496]
The Pólya urn-scheme is generalized to the case of balls of three colors. *W. Feller* (Ithaca, N. Y.).

Orts, Jose M.[a] The Legendre polynomials and the scheme of repeated trials. Revista Mat. Hisp.-Amer. (4) 1, 198–201 (1941). (Spanish) [MF 12998]
The author uses a property of Legendre polynomials to prove the (known) fact that in a series of n independent trials with a binomial population, the probability of the most probable number of successes approaches zero as $n\to\infty$. *D. Blackwell* (Washington, D. C.).

Fortet, R. Sur une suite également répartie. Rev. Sci. (Rev. Rose Illus.) 78, 298–299 (1940). [MF 13319]
This is a summary of results which have been published elsewhere [Studia Math. 9, 54–70 (1940); these Rev. 3, 169]. [Both the original paper and the summary contain a minor error which consists in asserting that if an increasing sequence of integers l_n is such that $l_n=O(n^\alpha)(\alpha>1)$, then $l_{n+1}-l_n$ is increasing.] *M. Kac* (Ithaca, N. Y.).

Feller, W. The fundamental limit theorems in probability. Bull. Amer. Math. Soc. 51, 800–832 (1945). [MF 14088]
A detailed exposition, including an exhaustive bibliography of developments connected with the central limit theorem and the law of the iterated logarithm.
J. L. Doob (Glen Echo, Md.).

Feller, W. Note on the law of large numbers and "fair" games. Ann. Math. Statistics 16, 301–304 (1945). [MF 13921]
The author points out that a game whose entry fee is the mathematical expectation of gain (supposed finite) may actually be very disadvantageous. He illustrates this by exhibiting such a game, in which the probability tends to one that after n trials the player will have sustained a net loss of order of magnitude $n(\log n)^{-\eta}$, where $\eta>0$ is arbitrarily small. In general, any practical theory of fairness must make the game entry fee depend on the number of trials to be made. As an example, the Petersburg paradox (in which the probability of winning 2^k is 2^{-k}) is analyzed. In this case the mathematical expectation of gain is infinite. It is shown that, if the player pays the amount $\log\log k$ at the kth trial, the following law of large numbers will hold: if S_n is the gain of the player in n trials, and if $P_n\sim n\log\log n$ is the sum of the first n entrance fees,

$$\lim_{n\to\infty}\text{prob}\{|S_n-P_n|<\epsilon P_n\}=1$$

for every $\epsilon>0$. This law of large numbers illustrates the reasonableness of varying entry fees. *J. L. Doob.*

Fan, Ky. Généralisations du théorème de M. Khintchine sur la validité de la loi des grands nombres pour les suites stationnaires de variables aléatoires. C. R. Acad. Sci. Paris 220, 102–104 (1945). [MF 13484]
The principal result announced is the following. Let $X_1, X_2, \cdots$ be a sequence of arbitrarily correlated random variables such that $E(X_n^2)$ are finite and

$$m^{-1}\{E(X_{n+1}+\cdots+X_{n+m})^2-E(X_1+\cdots+X_n)^2\},$$
$$E\{X_{m+1}(X_1+\cdots+X_m)\}-E\{X_{m+2}(X_1+\cdots+X_{m+1})\}$$

remain bounded for all m and n. Then $(X_1+\cdots+X_n)/n$ converges in probability. The proof is not given.
M. Kac (Ithaca, N. Y.).

Fan, Ky. Sur l'extension de la formule générale d'interpolation de M. Borel aux fonctions aléatoires. C. R. Acad. Sci. Paris 218, 260–262 (1944). [MF 13391]
The following theorems are stated in a slightly more general form. There exists a system of (ordinary) poly-

nomials $P_{m,n}(t)$, defined for $0 \leqq t \leqq 1$, $0 \leqq m \leqq n$, $n = 1, 2, \cdots$, such that for any random variable $X(t)$ which is continuous in probability and has a bounded expectation, and for every $\epsilon > 0$, one has

$$(*) \qquad \lim_{n \to \infty} \Pr\left\{ \left| X(t) - \sum_{m=0}^{n} X(m/n) P_{m,n}(t) \right| < \epsilon \right\} = 1,$$

uniformly in $0 \leqq t \leqq 1$. If to every ϵ there corresponds an η such that $|t - t_0| < \eta$ implies $E(|X(t) - X(t_0)|^r) < \epsilon$, then the sign Pr in (*) can be replaced by that of moment of rth order. *W. Feller* (Ithaca, N. Y.).

Fan, Ky. Sur l'approximation et l'intégration des fonctions aléatoires. Bull. Soc. Math. France **72**, 97–117 (1944). [MF 13222]

It is shown that a random function $X(t)$ $(0 \leqq t \leqq 1)$, continuous in the sense of Slutsky, can be approximated by polynomials in t with random coefficients. An analogue of approximation in the mean is discussed for a special class of random functions. Slutsky's definition of the integral of a random function is discussed and a descriptive definition of an integral proposed. *M. Kac* (Ithaca, N. Y.).

Blanc-Lapierre, André, et Brard, Roger. La loi forte des grands nombres pour les fonctions aléatoires stationnaires continues. C. R. Acad. Sci. Paris **220**, 134–136 (1945). [MF 13489]

Let the random variable $U(t)$ be stochastically continuous and let it have time-stationary moments of orders 1 and 2, and a stationary autocorrelation coefficient $\rho(\tau)$. Suppose that $\rho(\tau)$ is $(C, 1)$-summable over $(-\infty, \infty)$. Then $T^{-1}\int_0^T U(t)dt$ converges strongly to $E(U)$. This theorem is said to follow from known ergodic theorems if T takes on only values n^2 (n, an integer). The case of a continuously varying T can easily be reduced to that particular case. *W. Feller* (Ithaca, N. Y.).

Loève, Michel. Sur la covariance d'une fonction aléatoire. C. R. Acad. Sci. Paris **220**, 295–296 (1945). [MF 13942]

Let $X(\alpha)$ be a chance variable (with expectation 0) for each α in some set of real numbers. The author gives a systematic analysis of the properties of $X(\alpha)$ in terms of the correlation function $C(\alpha, \beta) = \text{expectation}\{X(\alpha)X(\beta)\}$. For example: (1) $X(\alpha)$ has a derivative $X'(\alpha)$ (the mean square limit of the difference quotient) if $C(\alpha, \beta)$ has continuous derivatives of the first two orders. The correlation function of $X'(\alpha)$ is then

$$\frac{\partial^2}{\partial\alpha\partial\beta} C(\alpha, \beta).$$

(2) The integral $\int_a^b X(\alpha)d\alpha$ is defined as the limit in the mean of the usual Riemann sums. Then

$$\text{l.i.m.}_{t \to \infty} \frac{1}{f(t)} \int_a^t X(\alpha)d\alpha$$

exists if and only if

$$\lim_{t \to \infty,\, t' \to \infty} \frac{1}{f(t)f(t')} \int_a^t \int_a^{t'} C(\alpha, \beta)d\alpha d\beta$$

exists. If $X(\alpha)$ is complex valued, $C(\alpha, \beta)$ is defined as the expectation of $X(\alpha)\overline{X(\beta)}$ and corresponding changes are made in the above results, where appropriate. Special cases, usually in the temporally homogeneous case, where $C(\alpha, \beta) = C(\beta - \alpha)$, have been commonly used by other writers. *J. L. Doob* (Glen Echo, Md.).

Loève, Michel. Analyse harmonique générale d'une fonction aléatoire. C. R. Acad. Sci. Paris **220**, 380–382 (1945). [MF 13950]

[See the preceding review.] The harmonic properties of $X(t)$ are related to those of the correlation function. For example,

$$X(\alpha) = \text{l.i.m.}_{T \to \infty} \int_{-T}^{T} e^{i\alpha\xi} \xi_t dt$$

(where ξ_t is a chance variable for each t) if and only if $C(\alpha, \beta)$ can be put in the form

$$C(\alpha, \beta) = \lim_{T \to \infty,\, T' \to \infty} \int_{-T}^{T} \int_{-T'}^{T'} e^{i(t\alpha - t'\beta)} \gamma(t, t')dtdt'.$$

If L_α is the operator $\sum_0^n a_m(\alpha)\partial^m/\partial\alpha^m$, and if L_α^* is the same operator with the coefficients replaced by their conjugates, $L_\alpha(X(\alpha)) = 0$ if and only if $L_\alpha L_\beta^* C(\alpha, \beta) = 0$. *J. L. Doob* (Glen Echo, Md.).

Dubrovsky, V. M. On purely discontinuous random processes with residual effect. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 79–81 (1945). [MF 13743]

The author considers a stochastic process which is not of the Markov type and where only a finite number of changes of state can occur in any finite time interval. Transition probabilities for an interval (t, τ) are expressed (by repeated integrations and summations) in terms of given elementary probabilities for changes at an arbitrary sequence of moments $t < t_1 < t_2 < \cdots < t_n < \tau$. Some numerical inequalities then follow easily. *W. Feller* (Ithaca, N. Y.).

Dedebant, G., et Wehrlé, Ph. Mécanique aléatoire. I. Le calcul aléatoire. Portugaliae Phys. **1**, 95–149 (1944). [MF 12938]

Elementary probability and statistics concern themselves with stochastic quantities (usually called random variables or variates). The parts of advanced probability and statistical mechanics which have to be invoked in a deeper study of such physical phenomena as Brownian movement, periodogram analysis, Markov chains, diffusion, turbulence and similar chaotic physical situations require the introduction of stochastic functions ("fonctions aléatoires"): X/t is defined as a stochastic function of the independent variable t if, when the value of t is given, a variate is determined. The essential thing is that, if n values $t_1, t_2, \cdots, t_n$ are given, n variates $X/t_1, X/t_2, \cdots, X/t_n$ are produced by this correspondence, and these variates are in general statistically interdependent. It is assumed that their joint distribution function, that is, that of the point $(X/t_1, \cdots, X/t_n)$ in n-space, is determined when the stochastic function X/t is given. A stochastic function is essentially a "random function" or random point in function-space. The original purpose of the authors was to write an extended treatise on the theory and applications of stochastic functions. The present paper forms the first part of a highly condensed outline of such a work, and it must be read as such. For example, many of the most important proofs and existence theorems have been omitted or reduced to skeletal form, and there are extremely few references.

After an introductory chapter, a discussion is given of the stochastic differential, that is, an expression for the stochastic function of two variables $Z/t, h = X/t + h - X/t$. The autocorrelation between $X/t + h$ and X/t plays a decisive role. This is followed by a consideration of the stochastic derivative $\dot{X}/t$, defined when a positive constant α exists

such that

$$\left(\frac{Z/t,\,h}{h^{\alpha/2}}-\dot{X}/t\right)^2$$

converges in the mean to zero as $h\to 0$, the convergence in the mean being with respect to the probability distribution in the 2-space $(X/t,\ X/t+h)$. Sufficient conditions are stated for the existence of $\dot{X}/t$. Successive derivatives are defined similarly in terms of higher order differences. All these notions are extended to stochastic functions of several variables and to sets of such functions (vector stochastic functions). Expansion of the correlation coefficients and the study of infinitesimal connections, vector correlations and tensor connections complete this chapter.

There follows a brief chapter on properties of the distribution function, in which differential equations, such as the equation of continuity, are shown to be satisfied by it. The following chapter, entitled Stochastic Integration, deals with the properties of various means, and the solution of equations connecting stochastic functions and their derivatives (stochastic differential equations). The statistical oscillator is studied as an application. The last chapter studies "stationary" stochastic functions. Three meanings are considered for this term: (a) X/t is stationary in probability when the simultaneous distribution function of X/t and $\dot{X}/t$ is independent of t; (b) X/t is simply stationary when the mean of the product $X/t_1\cdot X/t_2$ is a function of t_2-t_1 only; (c) X/t is completely stationary when for all n the mean of the product $X/t_1\cdots X/t_n$ depends only on the differences t_i-t_j. Harmonic analysis of the distribution functions of various sorts of stationary stochastic functions is introduced and the spectrum discussed. Periodicity is considered. *B. O. Koopman* (Washington, D. C.).

Bass, Jean, Dedebant, Georges, et Wehrlé, Philippe. Sur la connexion aléatoire d'un fluide. Application à la turbulence. C. R. Acad. Sci. Paris **220**, 165–167 (1945). [MF 13493]

Further discussion of the representation of a turbulent fluid as a single point moving in accordance with some probability law. [See Dedebant and Wehrlé, same C. R. **208**, 625–628 (1939)]. *J. L. Doob* (Glen Echo, Md.).

Bass, Jean. Sur la structure des fonctions aléatoires. C. R. Acad. Sci. Paris **220**, 190–192 (1945). [MF 13495]

[See the preceding review.] The author considers the differential equation satisfied by the probability density for position and velocity at time t in the following cases: (1) when the acceleration exists; (2) when the velocity $u(t)$ determines a Markov process. These cases are mutually exclusive. *J. L. Doob* (Glen Echo, Md.).

Bass, Jean. Quelques conséquences mécaniques de l'équation de structure d'un corpuscule aléatoire. C. R. Acad. Sci. Paris **200**, 272–274 (1945). [MF 13501]

[See the two preceding reviews.] The author considers further cases and finds that the differential equations do not give the full statistical description of the random motion; further statistical hypotheses (such as Markov properties) are necessary for unique determination. *J. L. Doob.*

Lévy, Paul. Sur le mouvement brownien dépendant de plusieurs paramètres. C. R. Acad. Sci. Paris **220**, 420–422 (1945). [MF 13953]

The time parameter of the ordinary homogeneous diffusion is replaced by a point A of the Euclidean space E_p. Let $r(A, B)$ denote the distance between the points A and B. A random variable $X(A)$ defines a "Brownian motion with p parameters" if (1) $X(A)-X(B)=\xi r^{\frac{1}{2}}(A, B)$, where ξ is a normalized Gaussian random variable; (2) the joint distribution of the variables $X(A_\nu)-X(0)$, $\nu=1, \cdots, n$, is an n-parameter Gaussian distribution. It is shown that such processes actually exist. Consider, for simplicity, the case $p=2$, and suppose the values of $X(A)=X(x, y)$ known for $y=0$. It is shown that the variables $X(0, y)-X(0, 0)$ and $X(0, 0)-X(0, -y)$ are not independent. Accordingly, the process is not of the Markov type, if this notion is generalized in the obvious way. *W. Feller* (Ithaca, N. Y.).

Wang, Ming Chen, and Uhlenbeck, G. E. On the theory of the Brownian motion. II. Rev. Modern Phys. **17**, 323–342 (1945). [MF 13698]

[Part I, by Uhlenbeck and Ornstein, appeared in Phys. Rev. (2) **36**, 823–841 (1930).] The authors give a general discussion of the random processes arising in the theoryof Brownian motions. The probability distribution of the spontaneous mesh charges in an arbitrary linear electrical network (that is, the Brownian motion of a system of coupled oscillators) is found explicitly by solving the corresponding Fokker-Planck differential equations. *J. L. Doob.*

Leroy, Robert, et Vaulot, Émile. Sur la proportion d'appels perdus dans certains systèmes de téléphonie automatique ne permettant dans un groupe d'organes qu'une seule exploration simultanée. C. R. Acad. Sci. Paris **220**, 84–85 (1945). [MF 13483]

The usual exchange system with n channels is considered under the hypothesis that the entire system is locked for a fixed period τ following each call. The steady state equations are written down and a "lost call formula" derived which reduces to Erlang's formula as $\tau\to 0$. *W. Feller.*

Mathematical Statistics

Burrau, Øyvind. The mean error as a measure of uncertainty. Mat. Tidsskr. B. **1943**, 9–16 (1943). (Danish) [MF 13592]

The author analyzes the shortcomings of the old-fashioned theory of the mean error and discusses the application of Student's theory to the theory of observations. *W. Feller* (Ithaca, N. Y.).

* **Dodd, Edward L. Lectures on Probability and Statistics.** University of Texas Press, Austin, Texas, 1945. 44 pp.

This is a memorial to the author, containing three lectures by him on various types of means. The commonly used averages, such as the arithmetic, geometric and harmonic means, the root-mean-square, the median and the mode, are emphasized. However, the main point of the lectures was to support the notion that substitution and not intermediacy is paramount in determining means; that is, the mean m may be expressed as $f(x_1, \cdots, x_n)$, where $f(t, \cdots, t)=f(t)$. Certain general properties of means were also included in the lectures. *R. L. Anderson* (Raleigh, N. C.).

Sarmanov, O. On isogeneous correlation. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 169–200 (1945). (Russian. English summary) [MF 13327]

The joint probability density function (p.d.f.) $F(x, y)$ of two random variables X, Y can be expressed in the forms

(1) $$F(x, y)=p(x)f_x(y)=P(y)\varphi_y(x),$$

where $p(x)$ and $P(y)$ are, respectively, the a priori p.d.f.'s of X and Y, and $f_x(y)$ and $\varphi_y(x)$ are the p.d.f.'s of Y given $X=x$ and of X given $Y=y$. If $f_x(y)$ and $\varphi_y(x)$ are both of the form

$$f_x(y)=\lambda(x)f(\lambda(x)y+\varphi(x)),$$
$$\varphi_y(x)=\lambda_1(y)f_1(\lambda_1(y)x+\varphi_1(y)), \tag{2}$$

X and Y are said to be isogeneously correlated [S. Bernstein, Metron 7, no. 2, 3–27 (1928)].

Substituting from (2) in (1) and eliminating the arbitrary functions, the author obtains a pair of fourth order differential equations which determine the most general forms for f and f_1. Three cases are considered separately, according to whether $F(x, y)$ is defined (a) over the whole (x, y)-plane, (b) over the half-plane $-\infty<x<\infty$, $b<y<\infty$, (c) over the quadrant $a<x<\infty$, $b<y<\infty$. Putting z equal to $\lambda(x)y+\varphi(x)$ and $\lambda_1(y)x+\varphi_1(y)$ in turn, and then $\Phi(z)=\log f(z)$, $\Phi_1(z)=\log f_1(z)$, the main results are combined in the following theorem. The only form of isogeneous correlation in cases (a), (b) or (c) is that in which $\Phi(z)$ and $\Phi_1(z)$ are logarithms of polynomials of degree at most two; in the limiting case, $\Phi(z)$ and $\Phi_1(z)$ may be polynomials of degree at most two.

Formulae are given for the regression lines and the variances of the x and y arrays. The particular cases $\varphi(x)\equiv\varphi(y)\equiv 0$ ("elastic correlation"), $\lambda(x)\equiv\lambda_1(y)\equiv 1$ ("stable correlation") and stable correlation with $\varphi(x)=-x(\rho\sigma_y/\sigma_x)$ (normal correlation) are studied in detail.

G. A. Barnard (London).

Bhattacharyya, A. A note on the distribution of the sum of chi-squares. Sankhyā 7, 27–28 (1945). [MF 13788]

The author first considers distributions of χ^2 for correlated normal variates x_1 and x_2, with zero means and variances σ_1^2 and σ_2^2. He finds that $\chi_1^2\pm\chi_2^2$ is distributed as a Bessel function and that $(\chi_1^2+\chi_2^2)/(\sigma_1^2+\sigma_2^2)$ is distributed as an infinite series of Laguerre polynomials. For $\chi^2=\sum_1^n x_k^2$, where each x_k has mean zero and variance σ_k^2 and the x's are independent, the distribution of $x=\chi^2/\sigma^2$ $(n\sigma^2=\sum\sigma_k^2)$ is also an infinite series of Laguerre polynomials:

$$\Gamma(\tfrac{1}{2}n)e^{-\frac{1}{2}x}x^{n-1}\sum_{r=0}^{\infty}d_rL_r(x, n)/\Gamma(n+r),$$

where d_r is a function of $(\sigma_k^2-\sigma^2)$. *R. L. Anderson.*

George, Aleyama. On the accuracy of the different approximations to the L_1-distribution. Sankhyā 7, 20–26 (1945). [MF 13787]

The accuracy of levels of significance given by approximations to the L_1-distribution is examined: $L_1=\lambda_1^{2/N}$, where λ_1 is the Wilks criterion for testing whether a set of k-samples belongs to p-variate normal populations with the same variance-covariance matrix [S. S. Wilks, Biometrika 24, 471–494 (1932)]. The exact distribution function of L_1 is given and probability curves sketched for some values of k and p ($k=2$, $p=2, 3, 4$; $k=3$, $p=2, 3$). Two approximations by Bishop are examined, and one each by Nayer (a type I curve), Bartlett and Pitman, the latter two only for $p=1$. In general, the size of sample must be increased to insure the same accuracy for an increasing number p of variables. The Bartlett and Pitman methods are shown to be almost alike and to give the best approximations to the true significance levels. The type I is definitely inferior for $p\geqq 3$. *R. L. Anderson* (Raleigh, N. C.).

Madow, William G. Note on the distribution of the serial correlation coefficient. Ann. Math. Statistics 16, 308–310 (1945). [MF 13923]

The distribution of the serial correlation coefficient, using the circular definition, is given for the population value $\rho\neq 0$. This is an extension of the results given by Anderson for $\rho=0$ [same Ann. 13, 1–13 (1942); these Rev. 4, 22]. *R. L. Anderson* (Raleigh, N. C.).

Barnard, G. A. A new test for 2×2 tables. Nature 156, 177 (1945). [MF 13285]

Wald, A. Sequential tests of statistical hypotheses. Ann. Math. Statistics 16, 117–186 (1945). [MF 12908]

A sequential test S of a statistical hypothesis H_0 is a test procedure which, at each stage of an experiment, gives a rule for making one of three decisions: (I) to accept H_0, (II) to reject H_0, (III) to continue the experiment by making an additional observation. If decision (I) or (II) is made at the nth stage, the procedure terminates at this stage; otherwise it continues.

Part I of the paper deals with sequential tests of a simple hypothesis H_0 against a single alternative H_1, the strength of a test S being specified by the probabilities (α, β) of rejecting H_0 when H_0 is true, and of accepting H_0 when H_1 is true. The successive stages of the experiment consist in successive observations of a random variable X. If n is the number of observations at which the procedure S terminates, n is a random variable whose distribution depends on that of X and on S. Let $E_i(n|S)$ denote the mean value of n if H_i is true $(i=0, 1)$. The ratio

$$(\text{g.l.b. max } \{E_i(n|S')\})/\max\{E_i(n|S)\},$$

where g.l.b. is taken over all S' of strength (α, β), is called the efficiency of S. If X is discrete (continuous), $f(x)$ denotes the probability (probability density) of the observation $X=x$. Hypothesis H_i says that f is f_i. If the first m observations are a_j $(j=1, \cdots, m)$, the probability ratio at the mth stage is $R_m=\prod_j\{f_1(a_j)/f_0(a_j)\}$. The sequential probability ratio test is obtained by choosing constants A, B, $0<B<A$, and making the rule at the mth stage: decision (I) if $R_m\leqq B$, (II) if $R_m\geqq A$, (III) if $B<R_m<A$. If $A=(1-\beta)/\alpha$ and $B=\beta/(1-\alpha)$, this test will in all practical cases have strength very nearly equal to (α, β).

The sequential probability ratio test is shown to have two remarkable advantages: (a) its efficiency is very nearly 1, and in fact both the $E_i(n|S)$ are nearly minimized; (b) its calculation requires nothing beyond a knowledge of f_i, so that no distribution problem is involved in setting it up. Thus in simplicity and efficiency it is markedly superior to the best classical tests based on fixed sample sizes. The efficiency of these latter is often below $\frac{1}{2}$. Using results of an earlier paper [same Ann. 15, 283–296 (1944); these Rev. 6, 88], a method is given for determining the probabilities of decisions (I) and (II) in case some hypothesis H, not necessarily an H_i, is true. Another method, illustrated for binomial and normal distributions, gives upper and lower bounds for these probabilities. Use of the earlier paper leads to the characteristic function, the moments, and the distribution of n (on a hypothesis H). Another method gives a lower bound for the probability that the procedure will terminate before a given stage. The effect of truncation, by excluding decision (III) at a predetermined stage, is also considered.

Part II deals with tests of simple or composite hypotheses against a set of alternatives. The case of a simple hypoth-

esis against one-sided alternatives is treated by straightforward generalization of part I, and is illustrated for binomial and normal distributions. Double dichotomies are reduced to the binomial case by an ingenious pairing of observations. In this section of the paper the exposition is elementary, for the benefit of prospective users who may lack background in mathematical statistics. Finally, the difficult problems of testing hypotheses with no restrictions on the alternative values of the unknown parameters are taken up. For simple hypotheses a solution is given which minimizes an arbitrarily weighted average of errors of the second kind. This is then extended to a partial solution for the case where the probability of error of the second kind should be less than a given value for any H whose "distance" from H_0 is greater than a given d. The case of k normal independent variates is given as example. Composite hypotheses are dealt with similarly. The analogue of the classical t-test situation is considered as an example.

G. A. Barnard (London).

Wald, Abraham. Sequential method of sampling for deciding between two courses of action. J. Amer. Statist. Assoc. **40**, 277–306 (1945). [MF 13412]

This is an expository paper applying the results of the paper reviewed above to sampling inspection problems where (a) the decision on a batch is restricted to acceptance or rejection, (b) the sample size (possibly variable) is always small compared with the size of the batch and (c) the quality of the batch and the sampling distribution involved are uniquely determined by a single parameter θ (the fraction defective, in the qualitative case). If $L(\theta)$, $E_\theta(n)$ denote, respectively, the probability of acceptance and the mean sample size, as functions of θ, the operating characteristic curve is the curve of $L(\theta)$ against θ and the average sample number curve is the curve of $E_\theta(n)$ against θ. The practical meaning of these curves is discussed. When the required operating characteristic curve is specified by two points on it, the sequential probability ratio test can be used to determine an inspection plan. Simple approximate formulae are given. The double dichotomy case, for determining which of two processes of production is preferable, is also discussed. *G. A. Barnard* (London).

Swan, A. W. Sampling schemes for qualitative inspection. Inst. Mech. Engrs. J. Proc. **152**, 81–92 (1945). [MF 12688]

The paper opens with an elementary discussion of sampling inspection and consumer's and producer's risks. The author then shows how to set up single and double sampling plans; this exposition is clear and illustrated with examples. Several tables and charts are provided which materially simplify calculations. *A. M. Mood* (Ames, Iowa).

Ville, Jean. Sur la théorie invariante de l'estimation statistique. Bull. Sci. Math. (2) **68**, 95–108 (1944). [MF 13254]

Let $\mathfrak{L}(P, M)$ be a probability density function depending on a parameter point P in an m-dimensional parameter space $[P]$ and a sample point M in n-dimensional sample space $[M]$. A method of estimation is determined by choosing a function $f(M|\mathfrak{L})$ depending on M and on the assumed form $\mathfrak{L}$ of the density function, the values of $f(M|\mathfrak{L})$ being point sets in $[P]$. If $\mathfrak{L}$ is assumed and M is observed, the statement is made that the unknown parameter point P is in $f(M|\mathfrak{L})$. Consider topological transformations $\mathfrak{T}_1$ of $[M]$ and $\mathfrak{T}_2$ of $[P]$; these generate a transform of $\mathfrak{L}$, say $\mathfrak{T}\mathfrak{L}$, namely, the probability density function of the transformed (vector) random variable $\mathfrak{T}_1M$, depending on the transformed parameter point $\mathfrak{T}_2P$. It is required that the method of estimation be defined also for $\mathfrak{T}\mathfrak{L}$, that is, that a function $f(\mathfrak{T}_1M|\mathfrak{T}\mathfrak{L})$ be defined whose values are point sets in $\mathfrak{T}_2[P]$. The method of estimation is said to be invariant if for all $\mathfrak{T}_1$ and $\mathfrak{T}_2$ we have $f(\mathfrak{T}_1M|\mathfrak{T}\mathfrak{L})=\mathfrak{T}_2f(M|\mathfrak{L})$. With reference to this concept several topics are touched on, including assignment of a sample to one of two completely specified populations, "distance" between two distribution functions, weight and risk functions, the maximum likelihood method of estimation. [Beware of misprints.] *H. Scheffé.*

Radhakrishna Rao, C. Generalisation of Markoff's theorem and tests of linear hypotheses. Sankhyā **7**, 9–16 (1945). [MF 13785]

Let $Y=(Y_1, \cdots, Y_n)$ be the vector of n chance variables and $\theta=(\theta_1, \cdots, \theta_n)$ the vector of their expectations. The covariance matrix of the Y's is assumed known to within a constant factor. It is given that $\theta=TA'$, where $T=(\tau_1, \cdots, \tau_m)$ is a vector whose m components are unknown, A is a known matrix with n rows and m columns, and an accent on a matrix denotes its transpose. Let L be a given vector. The problem of Markoff is, in this notation, to find a vector $B=(b_1, \cdots, b_n)$ such that (a) (the expectation) $E(BY')=LT'$ identically in T; (b) (the variance) $\sigma^2(BY')$ is a minimum. The generalization in the present paper is that no inequality relations between m and n and no restrictions on the rank of A are assumed. As a consequence, there exist vectors L for which no B satisfying (a) exists. A vector B exists when and only when there exists a vector Z such that $L=ZA$, and then $Z=B$. Uniqueness results are obtained and the normal equations are studied. Criteria for L's for which no unbiassed estimates exist are given. The material on tests of linear hypotheses is not new. *J. Wolfowitz.*

Radhakrisbna Rao, C. Markoff's theorem with linear restrictions on parameters. Sankhyā **7**, 16–19 (1945). [MF 13786]

The problem of the paper reviewed above is modified as follows. Let G be a known vector with k elements and R a known matrix with k rows and m columns. The problem now is to find a (not necessarily homogeneous) linear function b_0+BY' of the y's such that (a) $E(b_0+BY')=LT'$ identically for all T subject to $G=TR'$, (b) $\sigma^2(b_0+BY')$ is a minimum. It is proved that necessary and sufficient conditions for the existence of b_0 and B are the existence of an s and S such that, for a suitable D, $SA+DR=L$ and $s=DG'$. Then $b_0=s$ and $B=S$. Uniqueness conditions, existence criteria, and tests of hypotheses are discussed.

J. Wolfowitz (Raleigh, N. C.).

Tintner, Gerhard. A note on rank, multicollinearity and multiple regression. Ann. Math. Statistics **16**, 304–308 (1945). [MF 13922]

The author brings together several recent results in multivariate analysis in studying a problem in time series. A set of m variables $X_{it}=M_{it}+y_{it}$ $(i=1, \cdots, m; t=1, \cdots, n)$ are observed at n values of t; M_{it} is the systematic (trend and cycle) part of X_{it} and y_{it} is a disturbance or error normally distributed with zero mean. A test is provided for determining the number of linear relations among the M_{it}, the test criterion being a function of the roots of a determinantal equation. Given that there are R linear relations

$$k_{\nu 0}+\sum_i k_{\nu i}M_{it}=0, \qquad \nu=1, \cdots, R; t=1, \cdots, n,$$

among the M_i, the author also derives the maximum likelihood estimates of the coefficients k_r $(r=0, \cdots, m)$ under the assumption that the variance-covariance matrix (V_{ij}) of the y_i is known. These estimates would serve as approximations if (V_{ij}) were estimated by a large sample.

A. M. Mood (Ames, Iowa).

Gumbel, E. J. Simplified plotting of statistical observations. Trans. Amer. Geophys. Union **26**, 69–82 (1945). [MF 13524]

Smith, J. L. Spencer. The specification of disturbed periodic time series of the type of Wolfer's annual sunspot numbers. J. Roy. Statist. Soc. (N.S.) **107**, 231–241 (1944). [MF 13441]

The author represents the sunspot numbers as the sum of a sinusoid with variable phase and amplitude, and a variable trend. Most of the indeterminacy of this representation is removed by requiring high autocorrelations for the amplitude, the trend and, to a lesser extent, the phase. The treatment eliminates part of the effect of the autocorrelation of the original data in reducing the apparent period below its true value; the author obtains the (corrected) period 10.986 years. *A. Blake.*

Kishen, K. On the design of experiments for weighing and making other types of measurements. Ann. Math. Statistics **16**, 294–300 (1945). [MF 13920]

A partial solution is provided for the problem presented by Hotelling [same Ann. **15**, 297–306 (1944); these Rev. **6**, 93] of finding the best method of weighing p objects by N weighing operations. There are, in fact, two problems: one referring to the chemical balance and the other referring to the spring balance. The author deals only with the former and seems not fully aware that the two cases are distinct, since he points to some of his own designs for the chemical balance as superior to some of Hotelling's designs for the spring balance. He constructs weighing designs by employing a completely orthogonal matrix of order $N=2^m \geqq p$ whose elements are $+1$ or -1. These designs give estimates of the weights with smaller variances than any other designs using the same number of weighing operations. Other designs for which $N>2^m$ and $p \leqq 2^m$ are also discussed.

A. M. Mood (Ames, Iowa).

Jecklin, Heinrich. Über den Zusammenhang zwischen gewissen Zusatzversicherungen, Prämienzerlegungen und Approximationen in der Lebensversicherungstechnik. Mitt. Verein. Schweiz. Versich.-Math. **44**, 221–232 (1944). [MF 13962]

Zwinggi, Ernst. Über die Berechnung der unabhängigen Sterbe- und Stornowahrscheinlichkeiten im ersten Versicherungsjahr. Mitt. Verein. Schweiz. Versich.-Math. **45**, 57–66 (1945). [MF 13911]

Mathematical Biology

Wilkins, J. Ernest, Jr. The differential difference equation for epidemics. Bull. Math. Biophys. **7**, 149–150 (1945). [MF 12935]

The author shows that the equation of epidemics as studied by Wilson and Worcester [Proc. Nat. Acad. Sci. U. S. A. **30**, 37–44 (1944); these Rev. **6**, 12],

$$A-dS(t)/dt=BS(t)\{AC-S(t-T)+S(t-T-C)\},$$

where A, B, C, T are constants, has a stable equilibrium point $S=1/(BC)$ if $ABC>0$. *I. Opatowski.*

Wilson, Edwin B., and Worcester, Jane. Damping of epidemic waves. Proc. Nat. Acad. Sci. U. S. A. **31**, 294–298 (1945). [MF 13434]

The authors discuss the theory of epidemics on the basis of an equation of the form $A-S'(t)=(S/m)^p F$, where A, p and m are constants, S is the number of susceptibles, $S'=dS/dt$ and F may have three different expressions: (i) $p=1$, $F=A-S'(t-T)$, where T is a constant incubation time and the infective action is assumed to be instantaneous; (ii) p arbitrary, $F=A+S(t-T-G)-S(t-T)$, where G is the (constant) duration of the infective action; (iii) $p=1$,

$$F=\int_{-\infty}^{t}\{A-S'(\xi)\}\exp\{(\xi-t)/T\}d\xi$$

in the case of a time-distributed infective action according to Soper. The authors solve these equations by putting $S=m+\sum_{i=1}^{\infty}m_i y^i$ and neglecting terms with $i \geqq 2$. They show that the epidemic waves have a damping, which is in disagreement with an erroneous result of Soper in case (i) [J. Roy. Statist. Soc. N.S. **92**, 34–61 (1929)]. They point out also that replacement of derivatives by finite differences as done by Soper in case (iii) may lead to an error of 100% in calculating the damping coefficient.

I. Opatowski (Chicago, Ill.).

Morin, François, et Monod, Jacques. Sur l'expression analytique de la croissance des populations bactériennes. Revue Sci. (Rev. Rose Illus.) **80**, 227–229 (1942). [MF 13814]

The usual equation for the growth of a single population, limited only by the availability of nutrient material, has the form $dx/dt=x\mu(x)$, where x represents the population at time t. In the classical law of Verhulst $\mu(x)$ is linear. On a priori grounds $\mu(x)$ should not be concave upwards. The authors show by a simple geometric argument that the inflection of the curve of growth has an ordinate greater than half the ordinate of the asymptote when $\mu(x)$ is concave downward. They consider, for a nonreplenishable supply of nutriment, the case $dC/dt+Kdx/dt=0$, $\mu=\mu_0 C/(C_1+C)$, with which they integrate the equation of growth, reporting its agreement with experimental results and pointing out that experimentally one cannot distinguish between the analytical form of the resulting $\mu(x)$ and that proposed by other authors [for example, G. Teissier, Ann. Physiol. **10**, 359–376 (1934)] and possessing the same kind of convexity.

A. S. Householder (Washington, D. C.).

Rosenblatt, Alfred. On the application of mathematical statistics to bacteriology. I. Method of dilution of Louis Pasteur. Case of a single dilution. Application of the second law of P. S. Laplace. Univ. Nac. Tucumán. Revista A. **4**, 217–234 (1944). (Spanish) [MF 13025]

The author gives a method for estimating the parameter of a Poisson population. *D. Blackwell.*

García, Godofredo. On the integration of the equation of diffusion in the case of a spherical cell. Actas Acad. Ci. Lima **8**, 61–67 (1945). (Spanish) [MF 12723]

In a paper with the same title [Revista Ci., Lima **46**, 417–450 (1944) = same Actas **7**, 317–350 (1944); these Rev. **6**, 240] the author encountered the equation

$$\{f_1''+(2/r)f_1\}r^2+(C_0r^2-M)f_1=0.$$

Here he sets $f_1(r)=e^{\phi(r)}\lambda(r)$ and, considering only the case $\phi=0$, deduces recurrence relations among the coefficients of the series for $\lambda(r)r^{-\sigma}$, assumed analytic at the origin or at infinity. *A. S. Householder* (Washington, D. C.).

Lison, L. Analyse quantitative des facteurs de la morphogénèse des formes spirales logarithmiques biologiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **26**, 64–82 (1940). [MF 13829]

Consider an organism which grows by means of a secretion. The author studies various relations between the rate of local secretion and the rate of growth which would lead to logarithmic spirals in the organism's form. *W. Feller* (Ithaca, N. Y.).

Moon, Parry, and Spencer, Domina Eberle. Visual dark adaptation: A mathematical formulation. J. Math. Phys. Mass. Inst. Tech. **24**, 65–105 (1945). [MF 13350]

Various theories have been proposed for representing the dynamics of visual adaptation, and these have been accurate enough over limited domains. But seldom, if ever, is it possible to carry over any of the parameters from one set of experimental conditions to another, with the result that while there are in the literature many sets of data, each lying accurately along a rationally deduced curve, the curves themselves are quite unrelated.

The authors have taken a long step toward the unification which is to be desired. They start with the well-known fact that in the rods rhodopsin breaks down under the influence of light into retinene, but they depart from previous theorists in supposing that the energy-absorbing breakdown sets off a sequence of two or more dark reactions, the velocity of the final reaction being proportional to an exponential in the concentration of the retinene, rather than a linear function. This leads to the first part of the right member of

$$dx/dt=k_1H(\varepsilon^{-at}-\varepsilon^{-bt})-\left[k_2-k_4x^2+A\exp\left\{-G\int_{-\infty}^{t}\frac{ak_1H(s)\varepsilon^{-b(t-s)}}{1+ak_1sH(s)}ds\right\}\right]x.$$

The quantity in brackets, representing the rate of transformation of retinene back to rhodopsin, is deduced largely ad hoc by curve-fitting, except that the integral is based upon the reasonable assumption that the rate of the transformation is affected by the entire previous exposure ($H(s)$ is the brightness at time s) but less markedly at more remote times. To relate x to measurable variables it is assumed that each degenerating rhodopsin molecule releases a single electron which sets off a single nervous impulse (or some fixed number), and that the threshold stimulation represents a fixed minimal frequency of these impulses. *A. S. Householder* (Washington, D. C.).

TOPOLOGY

Colmez, Jean. Discussion préliminaire du problème de Wiener. Revue Sci. (Rev. Rose Illus.) **80**, 313–315 (1942). [MF 13811]

The problem is to determine a topology for an infinite set S such that each member of a preassigned group G of permutations of S (one-to-one correspondences of S with itself) becomes a homeomorphism of S. There are obvious solutions in which all finite sets, but not all subsets, of S are closed. The general solution is given for the case in which G includes all permutations of S. *R. Arens.*

Ridder, J. Über topologische Eigenschaften von Strukturen. Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **18**, no. 4, 43 pp. (1944). [MF 13710]

The axiomatic investigation of structures L (lattices, in the terminology of G. Birkhoff) is carried out by admitting systematically not only axioms of a lattice-theoretic nature, but also postulates regarding the "open" elements $g\in L$. The choice and formulation of these postulates is guided by a consideration of lattices of subsets of topological spaces satisfying various familiar conditions. The emphasis is on equivalence, independence, etc. of axiom systems for various types of lattices in which "open" elements are distinguished. *R. Arens* (Princeton, N. J.).

Szumbarski, M. Sur la décomposition des eléments euclidiens en produits cartésiens. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 39–42 (1945). (French. Russian summary) [MF 13011]

The spaces X and Y are said to be divisors of every space Z homeomorphic to the product $X\times Y$. The author proves two theorems. (1) A simple arc is the only 1-dimensional divisor of Euclidean n-space, $n\geqq 1$. (2) Every divisor of Euclidean n-space, $n\leqq 3$, is itself a Euclidean space. The proof of (1) proceeds by showing that the only 1-dimensional divisor of a Euclidean n-space is a dendrite without ramification point, and hence a simple arc. The proof of (2) is carried out by separate consideration of several cases. *H. Wallman* (Cambridge, Mass.).

Dieudonné, Jean. Une généralisation des espaces compacts. J. Math. Pures Appl. (9) **23**, 65–76 (1944). [MF 13797]

Paracompact spaces are topological spaces in which any covering by open sets can be replaced by a neighborhood-finite refinement. (A covering Σ is a refinement of a covering Σ^* if each member of the covering Σ is contained in some member of Σ^*; Σ is neighborhood-finite if each point of the space has a neighborhood meeting only a finite set of sets of Σ.) A paracompact Hausdorff space is normal, and a separable metric space is paracompact. If A is compact and B is paracompact, then $A\times B$ is paracompact; but it is unknown whether $B\times B$ must be paracompact, or whether all metric spaces are paracompact. *R. Arens.*

***van Heemert, Anthonie. De R_n-adische Voortbrenging van Algemeen-Topologische Ruimten met Toepassingen op de Constructie van niet Splitsbare Continua. [The R_n-adic Valuation of General Topological Spaces with Applications to the Construction of Indecomposable Continua].** Thesis, University of Groningen, 1943. 188 pp. (Dutch)

Zunächst behandelt der Verfasser die Grundlagen der allgemeinen Topologie im Anschluss an Alexandroff-Hopf, Topologie I. [Springer, Berlin, 1935]. Er bemerkt dabei einige kleine Verschärfungen, z.B. dass die Menge der Häufungspunkte einer Teilmenge eines kompakten topologischen Raumes R kompakt ist, wenn in R das erste Trennungsaxiom gilt, ferner dass das stetigen Bild eines kompakten topologischen Raumes unter denselben Bedingungen kompakt ist. Die R_n-adische Entwicklung von Räumen setzt der Verfasser allgemeiner als Freudenthal

[siehe z.B. Compositio Math. 4, 145–235 (1937)] an, nämlich für ganz allgemeine topologische Räume, nicht nur für die Kuratowskischen Räume. Abgesehen von den auch bei Freudenthal bewiesenen Eigenschaften beweist der Verfasser noch einen Permanenzsatz der Relativierung, der besagt, dass die Relativierung der topologischen Zuordnung und die R_α-adische Limes-Bildung miteinander vertauschbar sind. Ganz ausführlich beschäftigt der Verfasser sich mit der Permanenz von allerlei topologischen Axiomen. Die Begriffe Umgebungsraum, metrischer Raum, allgemein-metrischer Raum und topologischer Raum sind invariant beim R_α-adischen Grenzübergang, für den Begriff Konvergenzraum gilt das nur beim Vorhandsein einer abzählbaren lokalen Basis. Gültigkeit der Trennungsaxiome ist gleichfalls eine invariante Eigenschaft; nur für das vierte Trennungsaxiom konnte der Verfasser die Frage nicht entscheiden. Kompaktheit ist bei einfachen Zusatzvoraussetzungen invariant; für die lokale Kompaktheit gilt das nicht. Ein ähnliches Verhältnis findet man bei dem Zusammenhang und dem lokalen Zusammenhang.

Nachdem der Verfasser noch auf die R_α-adische Polyeder-Entwicklung von Freudenthal eingegangen ist, kommt er zu erheblich origineller fundierten Resultaten. Der Verfasser selbst hatte bereits früher [Compositio Math. 5, 319–326 (1937)] die Behauptung von Freudenthal, dass die von D. van Dantzig [Fund. Math. 15, 102–125 (1930)] entdeckten eindimensionalen Solenoiden unzerlegbare Kontinua sind, mittels der R_α-adischen Methode bewiesen, wobei auch die Konstruktion der Solenoiden eben auf der R_α-adischen Methode beruhte. Er verwendet nun ganz allgemein die R_α-adische Methode zur Konstruktion unzerlegbarer Kontinua. Er nennt eine Abbildung $fA=B$ eines Kontinuums in ein anderes unzerlegbar, wenn bei jeder Zerlegung von A in zwei Kontinua C und D entweder $fC=B$ oder $fD=B$ gilt. Abbildungen von Strecken aufeinander kann man leicht ansehen, ob sie unzerlegbar sind. Nun zeigt sich, dass R_α-adische Folgen mit unzerlegbaren Abbildungen zu unzerlegbaren Limes-Kontinua führen. Sind die R_α Strecken, so erhält man als Limites sogenannte s-Kontinua, die absolut-unikohärent sind, und diese s-Kontinua sind unzerlegbar, sobald die erzeugenden Abbildungen es sind. Das ist ein sehr einfaches Erzeugungsverfahren, das die meisten bekannten unzerlegbaren Kontinua liefert. Und zwar kommen auch die Knasterschen absolut-unzerlegbaren Kontinua heraus, sobald man den Begriff der unzerlegbaren Abbildung noch in geeigneter Weise verfeinert. Statt der Strecken als erzeugende Räume R_α kann man aber auch allgemeiner irreduzible Kontinua wählen. Ausgehend von der Kuratowskischen Theorie [Fund. Math. 10, 225–275 (1927)] definiert der Verfasser die schichtenweise Abbildung von irreduziblen Kontinua als Abbildung, bei der die Schichten ("tranches") erhalten bleiben. Besteht eine R_α-adische Erzeugung von R durch die Kontinua R_α aus schichtenweisen Abbildungen, sind ferner die Anfangs- und Endschichten jedes R_α Kondensationskontinua oder Punkte und ist das R zugeordenete s-Kontinuum der Schichten unzerlegbar, so ist auch R unzerlegbar. So kommt der Verfasser zu unzerlegbaren Kontinua von beliebiger Dimension und zwar auf viel einfachere Weise als Urysohn, von dem das einzige Beispiel in dieser Richtung stammte. Auch ganz neue Typen absolut unzerlegbarer Kontinua ergeben sich so. Das Knastersche Problem der Existenz absolut unzerlegbarer Kontinua von höherer als der ersten Dimension bleibt jedoch immer noch ungelöst.

Zum Schluss behandelt der Verfasser noch die Unzerlegbarkeit der Solenoiden und entwickelt weitere Konstruktionsmethoden für unzerlegbare Kontinua, die die R_α-adische Erzeugung der Solenoiden verallgemeinern.

H. Freudenthal (Amsterdam).

*de Groot, Johannes. **Topologische Studien. Compactificatie, Voortzetting van Afbeeldingen en Samenhang. [Topological Studies. Compactification, Extension of Mappings and Connectivity].** Thesis, University of Groningen, 1942. 102 pp. (Dutch)

Der erste Hauptsatz des Verfassers ist, dass ein Hausdorffscher Raum M, der in den Punkten einer kompakten Menge N im Kleinen semikompakt, aber sonst überall im Kleinen kompakt ist, sich durch ein nulldimensionale Menge zu einem kompakten Hausdorffschen Raum ergänzen lässt. Keinerlei Abzählbarkeitsaxiome usw. werden vorausgesetzt.

Dieser Satz ist einigermassen isoliert. Übrigens ist die Dissertation einem einheitlichen Fragenkomplex aus der Kompaktifizierungstheorie gewidmet. Zunächst gilt der Hauptsatz: Unter den separablen Räumen sind die semibikompakten identisch mit den nulldimensional kompaktifizierbaren (d.h. den Räumen, die durch nulldimensionale Mengen zu kompakten separablen Räumen ergänzt werden können). Dieser Satz steht implizit auch bei Freudenthal [Ann. of Math. (2) 43, 261–279 (1942); diese Rev. 3, 315]. Auch den sich mit diesem überschneidenden Satz von Zippin [Amer. J. Math. 57, 327–341 (1935)] über die Kompaktifizierbarkeit semikompakter vollständiger Räume durch abzählbare Mengen beweist der Verfasser hier von Neuem.

Die Betrachtung der semikompakten Räume führt auf natürliche Weise zur Freudenthalschen Endentheorie. Als Vorbereitung entwickelt der Verfasser Zusammenhangsbegriffe. Zwei Punktmengen eines Raumes heissen quasiverbunden, wenn jedes Stück (d.h. offene und abgeschlossene Teilmenge), das mit der einen einen Punkt gemein hat, auch mit der andern einen Punkt gemein hat. Auf diesem Begriff beruht der (bekannte) Begriff der Quasikomponente. Nennt man einen normalen Raum quasibikompakt, wenn je zwei abgeschlossene elementefremde Mengen beliebig kleine elementefremde Umgebungen mit bikompakten Rändern besitzen, so gilt der Satz: In quasibikompakten Räumen fallen Komponenten- und Quasikomponenten-Raum zusammen; sie sind dann nulldimensional. In allgemeinen Räumen braucht diese Identität bekanntlich nicht zu gelten.

Der Begriff des Quasizusammenhang hat sich als ausschlaggebend erweisen bei den Untersuchungen des Verfassers über die Fortsetzung topologischer Abbildungen. Er beweist u.a. folgenden Satz. Gegeben die Kompakta $A+B$ und $A'+B'$; A und A' mögen homöomorph sein, $B\subset\bar{A}$, $B'\subset\bar{A}'$, $\dim B=\dim B'=0$. Ist der Quasikomponenten-Raum von A kompakt, und gibt es in jeder Umgebung (rel. A) eines Punktes b von B nur eine Quasikomponente, die sich bei b häuft, so ist jede topologische Abbildung von A auf A' fortsetzbar zu einer topologischen Abbildung von $A+B$ auf $A'+B'$. Dieser gründliche Erweiterungssatz steht in engem Zusammenhang mit der Endentheorie, die der Verfasser durch die axiomatische Forderung der "idealen Kompaktifizierung" erzwingt. Er nennt den separablen Raum M ideal kompaktifizierbar, wenn eine nulldimensionale Menge N existiert, so dass $\bar{M}=M+N$ mit einem maximalen N gilt (d.h. ist N' eine andere kompaktifizierende nulldimensionale Menge, so ist jede topologische Abbildung von M auf sich selber zu einer stetigen Abbildung von $M+N$ auf $M+N'$ fortsetztbar). Es zeigt sich, dass die ideale Kompaktifizierung (wenn sie existiert) mit der durch

die Endpunkte identisch ist, und dass sie unter denselben Bedingungen wie die Letztere existiert. Die ideal kompaktifizierbaren Räume sind mit den Semikompakta identisch, die einen kompakten Quasikomponenten-Raum besitzen. Der Beweis ist unabhängig vom Freudenthalschen.

H. Freudenthal (Amsterdam).

Bing, R. H. Collections filling up a simple plane web. Bull. Amer. Math. Soc. 51, 674–679 (1945). [MF 13602]

In what follows, G and H denote two distinct upper semicontinuous decompositions into continua such that each element of G meets each element of H in a 0-dimensional set, the decomposition spaces being nondegenerate. In accordance with the first theorem a simple plane web may be defined as a compact plane continuum admitting a pair G, H of such decompositions. The author shows that, if W is a simple plane web and G, H is such a pair of decompositions with dendritic hyperspaces, then G has an element which does not separate W and which is not contained in the closure of a connected set which is the union of a countable number of points and complementary domains of W. Also, with the above hypotheses and the additional assumption that W has two complementary domains, G is not equicontinuous. In other words, there exists a positive number ϵ such that for each $\sigma>0$ an element g of G can be found containing points $p\neq q$ for which $\rho(p, q)<\sigma$ but such that there is in g no continuum of diameter less than ϵ which contains both p and q. The last theorem states that if, moreover, G is an arc with respect to its elements then W contains a point at which G is hereditarily nonequicontinuous.

A. D. Wallace (Philadelphia, Pa.).

Whyburn, G. T. Extensions of plane continua mappings. Amer. J. Math. 67, 505–520 (1945). [MF 13938]

The author is concerned with extending the domain of definition of a transformation so as to maintain other properties in addition to continuity. The sets considered are usually locally connected continua imbedded in a sphere, and the given transformations are extended to the whole sphere. The transformations considered are monotone, nonalternating, interior or light, or a combination of these types. The following two theorems are representative of the results. (1) Any nonalternating mapping $f(A)=D$ of a locally connected continuum on a sphere onto a dendrite D is monotonically extensible to S. (2) Any light interior and nonalternating transformation $f(A)=ab$ of a locally connected continuum A on a sphere S onto a simple arc ab can be extended to a monotone interior mapping $\phi(S)=ab$. Two examples of interesting light interior mappings of a θ-curve are given.

J. H. Roberts (Durham, N. C.).

Wallace, A. D. Dimensional types. Bull. Amer. Math. Soc. 51, 679–681 (1945). [MF 13603]

The author makes the following definition: if H and S are topological spaces, H is said to be of dimensional type S if for each closed subset X of H and mapping $f:X\to S$ there exists an extension $\hat{f}:H\to S$. Established results show that if H is separable metric and S_n an n-sphere then H has dimensional type S_n if and only if $\dim H\leqq n$ in the classical sense; hence for separable metric spaces there is equivalence between the classical dimension and the least n for which the space is of dimensional type S_n. For more general spaces it is shown that (a) if a certain separation axiom is satisfied then having dimensional type S_n implies having dimensional type S_{n+1}; (b) (sum theorem) if H is normal, S an absolute neighborhood retract, and $H=H_1+H_2+\cdots$, with each H_i closed and of dimensional type S, then H has dimensional type S.

H. Wallman (Cambridge, Mass.).

Shkliarsky, D. On subdivisions of the two-dimensional sphere. Rec. Math. [Mat. Sbornik] N.S. 16(58), 125–128 (1945). (Russian. English summary) [MF 13006]

The paper is concerned with closed finite coverings $\{f_i\}$ of the two-dimensional sphere S. The author shows that (1) if $\{f_i\}$ has the property that there exists a homeomorphism of S for which the sets f_i are disjoint from their images, then at least three of the f_i intersect. From this it immediately follows that (2) if S is covered by three closed sets f_1, f_2, f_3, then, given an arbitrary homeomorphism of S [the author's English summary confuses matters by employing the word deformation], there is a point $x\in S$ such that both x and its image belong to the same f_i. From (2) follows the theorem of Schnirelman: if S is covered by three closed sets then at least one must contain a pair of antipodal points.

H. Wallman (Cambridge, Mass.).

Smith, P. A. Transformations of finite period. IV. Dimensional parity. Ann. of Math. (2) 46, 357–364 (1945). [MF 13423]

[The first three parts appeared in the same Ann. (2) 39, 127–164 (1938); 40, 690–711 (1939); 42, 446–458 (1941); these Rev. 1, 30; 2, 324.] Let T be a topological transformation of finite period of a compact space R into itself. Cocycles, etc. are in the sense of Čech. Let L be the set of fixed points; let r be its dimension. Theorem: if p is an odd prime and R has certain cohomology properties like those of an n-sphere ($n>0$), then $n-r$ is even. The same result holds for an Abelian transformation group of order p^a (p an odd prime) if R is as before, and R and L have a certain locally Euclidean character. It is stated that the results hold with trivial modifications if R is merely assumed to possess the above properties locally.

H. Whitney.

Laboureur, Jacques. Propriétés topologiques du groupe des automorphismes de la sphère S^n. Bull. Soc. Math. France 71, 206–211 (1943). [MF 13236]

Let Ω_{n+1} be the group of rotations (determinant $+1$) of the n-sphere S^n, and let G_n be the group of automorphisms of S^n (the group of homeomorphisms of S^n with itself of degree $+1$). Let G_n^0 be the isotropy group of G_n (subgroup leaving a point fixed); G_n^0 has G_{n-1} as subgroup in a natural way. The author proposes to prove that G_{n-1} is a deformation retract of G_n^0. The argument runs as follows. Denote by Γ_n the group of all automorphisms of an n-cell C^n: $\sum_1^n x_i^2\leqq 1$. Alexander has given a retraction of Γ_n on the subgroup Γ_n' of "radial" automorphisms which leave the distance from the origin invariant and map each radius on a radius; Γ_n' is isomorphic with G_{n-1}. Now with the help of the well-known mapping j of C^n on S^n, by identifying the boundary of C^n into a point, Alexander's retraction is "translated" into a retraction of G_n^0 onto G_{n-1}. It seems to the reviewer that this is not valid as it stands, since j maps Γ_n not onto all of G_n^0, but on a proper subgroup only (for $n>1$); mappings of the n-cell, which are topological in the interior only, but not on the boundary, become perfectly good members of G_n^0 under j.

The remaining theorems, whose proofs depend on the above theorem, state: Ω_{n+1} is a deformation retract of G_n; the homogeneous space G_n/Ω_{n+1} is contractible and has a cross section in G_n. This would explain the rôle played by Ω_{n+1} in the theory of sphere bundles.

H. Samelson.

Nielsen, Jakob. Abbildungsklassen endlicher Ordnung. Acta Math. 75, 23–115 (1943). [MF 13205]

Consider the orientation-preserving homeomorphisms of a compact, connected, orientable, 2-dimensional manifold φ onto itself. The author proves the following theorem. If the nth iterate k^n of such a homeomorphism k is homotopic to the identity homeomorphism then k itself is homotopic to a homeomorphism k' whose nth iterate is precisely the identity. For the manifolds φ of nonnegative Euler characteristic (the sphere, 2-cell, circular ring and torus) the theorem is more or less trivial because the mapping classes which satisfy the hypothesis of the theorem can be considered individually. If φ has a negative Euler characteristic then a metric of constant negative curvature can be introduced in such a way that the boundary curves, if there are any, are geodesics. The universal covering surface can be represented as the "convex figure" K_F of a non-Abelian group F of hyperbolic linear fractional transformations which leave the unit circle E and its interior D set-wise unaltered [cf. the author's paper in Mitt. Math. Ges. Hamburg 8, part 2, 82–104 (1940); these Rev. 2, 213]. The homeomorphism k determines a homeomorphism κ of K_F which in turn determines a homeomorphism I of the limit set $\bar{G}_F = \bar{K}_F \cap E$. Since any two points of $\bar{G}_F$ determine a circle orthogonal to E and any such circle covers a geodesic in φ, the mapping k determines a "permutation" K of the set of geodesics of φ. The mapping I and the permutation K are of order n, are unaltered by homotopy of k, and determine k up to homotopy. It is possible to introduce a geodesic reticulation of φ which is invariant under the permutation and is such that the boundary of any 2-cell corresponds to itself under the mth iterate of K only when $m \equiv 0 \pmod{n}$. A homeomorphism k' of φ onto itself, which realizes this permutation of the cells of the reticulation, can then be constructed, first over the 1-cells and then over the 2-cells. The whole construction, which is rather complicated, is made in K_F and not in φ as indicated here.

R. H. Fox (Princeton, N. J.).

Nielsen, Jakob. A theorem on the topology of surface transformations. Norsk. Mat. Tidsskr. 23, 5 pp. (1941). (Danish) [MF 13772]

Discussion of the significance of the main theorem of the paper reviewed above. *R. H. Fox* (Princeton, N. J.).

Nielsen, Jakob. Fixed point free mappings. Mat. Tidsskr. B. 1942, 25–41 (1942). (Danish) [MF 13587]

Consider the orientation-preserving topological transformations of a closed two-sided surface of genus p onto itself which, together with a certain number of their iterates, are fixed point free. By the degree of fixed point freedom (designated below simply as the degree) of such a transformation is meant the least integer t such that the tth iterate has at least one fixed point. If $p>2$ the maximum degree of a transformation is $2(p-1)$. If $p=2$ the maximum degree is either 2 or 3: just which of these two numbers is the exact maximum is an unsolved problem. If consideration is restricted to those transformations which are periodic (but of arbitrary period), then the maximum degree is $p-1$ for $p>2$ and 2 for $p=2$. (A moment's reflection shows that these two problems are easily settled for $p=0$ and $p=1$.) These assertions are proved from the characteristic polynomial of the induced automorphism of the 1-dimensional homology group, making use of the Alexander trace-formula, the modular surface of a periodic mapping, the Riemann-Hurwitz branch-point formula and Wiman's inequality, and suitably chosen examples. There is some discussion of the eccentric case $p=2$. Finally, it is shown that a (nonperiodic) transformation of maximum degree $2(p-1)$ leaves unaltered the type of a certain curve of the surface, and that, on the bounded manifold obtained by cutting along such a curve, a periodic transformation is necessarily induced. *R. H. Fox* (Princeton, N. J.).

Hirsch, Guy. Quelques théorèmes sur les points fixes des groupes de représentations. Bull. Soc. Roy. Sci. Liége 12, 392–407 (1943). [MF 13150]

In an earlier paper [Ann. Sci. École Norm. Sup. (3) 60, 113–142 (1943)] the author obtained a number of fixed-point theorems of which the following is typical. Let T be a mapping of a complex C onto itself, permutable with the elements of a transformation group **A**. Let **B** be the subgroup generated by the transformations of **A** whose fixed-point sets have nonzero intersections with the fixed-point set of T. Then the Lefschetz number of T is a multiple of the index of **B** in **A**. The present paper contains a number of consequences of these theorems with particular reference to periodic transformations. For example, let A be a periodic transformation operating in C and assume that no power of A except the identity admits fixed points. Then the period a of A is a divisor of the Euler characteristic $e(C)$. It follows, of course, that if $e(C) \neq 0$, a cannot be arbitrarily large. For a given prime p and arbitrary nonnegative m, the Lefschetz number of A^{mp} is divisible by p; this remains true if the assumption of periodicity is dropped. Results of this sort were known previously only for surface transformations and were obtained (notably by Nielsen) by quite different methods. *P. A. Smith* (New York, N. Y.).

Hirsch, Guy. Sur les groupes d'homologie de certains complexes de recouvrement. Portugaliae Math. 4, 225–237 (1945). [MF 13362]

Relations are obtained between the homology groups of a complex C and a regular covering $\bar{C}$ of C of finite order f. By factoring the homology groups H_r, $\bar{H}_r$ by torsion subgroups whose orders divide f, groups H_r', $\bar{H}_r'$ are obtained which are proved to be isomorphic if each transformation of the covering group induces the identity transformation in $\bar{H}_r$. Under this assumption, the rth Betti numbers of C and $\bar{C}$ are equal; in addition, H_r, $\bar{H}_r$ are isomorphic if coefficients modulo m are used, where m is prime to f. Applications are made to coverings of Lie groups and to transformation groups of spheres. *N. E. Steenrod.*

de Kerékjártó, B. Sur le caractère topologique du groupe homographique de la sphère. Acta Math. 74, 311–341 (1941). [MF 13771]

This paper is identical with one already reviewed [J. Math. Pures Appl. (9) 21, 67–100 (1942); these Rev. 5, 60], except for the following additional remark. The plane and the sphere are the only surfaces which admit doubly or triply transitive groups of transformations.

H. Samelson (Syracuse, N. Y.).

Eilenberg, Samuel, and MacLane, Saunders. Relations between homology and homotopy groups of spaces. Ann. of Math. (2) 46, 480–509 (1945). [MF 13433]

The formulas obtained in this paper generalize and include all earlier known relations between the homotopy and homology groups of an arcwise connected space X. If

the homotopy groups $\pi_i(X)$ vanish for $i>1$, the homology groups $H^n(X, G)$ are determined by $\pi_1(X)$. Explicitly,

(1) $$H^n(X, G) \cong H^n(K(\pi_1(X)), G).$$

Here K is a complex constructed from the group $\pi_1(X)$ as follows. An h-cell of K is any $(h+1)$-rowed square matrix $\|d_{ik}\|$ of elements of $\pi_1(X)$ represented by the edges of a singular h-cell in X whose vertices all coincide with a fixed base point x_0. The boundary cells are the h-rowed matrices obtained by striking out single rows and their transposes. If $\pi_n(X)=0$ for $1<n<r$, then for these values of n (1) holds. In addition

(2) $$H^r(X, G)/\Sigma^r(X, G) \cong H^r(K(\pi_1(X)), G),$$

(3) $$\Lambda_r(X, G) \cong H_r(K(\pi_1(X)), G),$$

where Σ is defined with the aid of r-dimensional spherical cycles and Λ_r is a certain annihilator of Σ^r.

It is stated that (1) also holds when all homotopy groups except one, say $\pi_m(X)$, vanish. In this case, however, $K(\pi_1(X))$ is to be replaced by $K(\pi_m(X))$ the construction of which, for $m>1$, is necessarily different from the construction for $m=1$. Similar generalizations are stated for (2) and (3). *P. A. Smith* (New York, N. Y.).

Cartan, Henri. Méthodes modernes en topologie algébrique. Comment. Math. Helv. **18**, 1–15 (1945).

If E is locally compact, $\hat{E}$ is the compact extension of E obtained by adjoining a point at infinity. The author defines the Lefschetz groups of (E, F), where F is closed in E, to be the Čech homology groups of $(\hat{E}, \hat{F})$. Using these groups, the author deduces classical results such as the invariance of domain, of dimension, the Jordan-Brouwer separation theorem, etc. The interest lies chiefly in the methods used. Once the general properties of the Lefschetz groups are established, all subsequent results are deduced from these properties alone, thus avoiding any additional use of complexes or assumptions of triangulability. These basic properties are almost exactly the ones used by Eilenberg and Steenrod in their axiomatic development of homology theory [Proc. Nat. Acad. Sci. U. S. A. **31**, 117–120 (1945); these Rev. **6**, 279]. *N. E. Steenrod.*

*Cartan, Henri. **Méthodes modernes en topologie algébrique.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, p. 246, Füssli, Zürich, 1945.
Summary of the paper reviewed above.

Ehresmann, Charles. Sur les applications continues d'un espace dans un espace fibré ou dans un revêtement. Bull. Soc. Math. France **72**, 27–54 (1944). [MF 13217]

The covering homotopy theorem (c.h.t.) for fibre spaces, which was first proved by Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U. S. A. **27**, 61–64 (1941); these Rev. **2**, 323], proved independently by Eckmann [Comment. Math. Helv. **14**, 141–192 (1942); these Rev. **3**, 317], and announced independently by Ehresmann and Feldbau [C. R. Acad. Sci. Paris **212**, 945–948 (1941); these Rev. **3**, 58], is here proved, using a finite complex for antecedent space and using a definition of fibre space which is less general than the definition of Hurewicz and Steenrod. Using Zorn's lemma, the c.h.t. is then generalized to allow the antecedent to be an infinite complex. [The proof of this generalization applies without any alteration to the general c.h.t. of Hurewicz and Steenrod.] It should be noted that the author's formulation of the c.h.t. as a theorem on the existence of an extension of a partly defined covering homotopy is an easy consequence of the c.h.t. and was proved by Steenrod [Ann. of Math. (2) **43**, 116–131 (1942); these Rev. **3**, 144].

The bulk of the paper is concerned with covering spaces, which are defined as fibre spaces which have discrete fibres. The standard properties of covering spaces are derived from this definition. The reviewer observes that proposition 4 has been proved (under more restrictive conditions) by Franz [J. Reine Angew. Math. **185**, 65–77 (1943); these Rev. **5**, 103]. Theorem 3, which gives necessary and sufficient conditions for the homotopy of mappings into a locally Euclidean or locally hyperbolic space, seems to be new. In an appendix a theorem of Hurewicz about the classification of mappings into an aspherical space is proved under more general conditions. [It should be remarked that most of the papers quoted in this review were probably inaccessible to the author.] *R. H. Fox* (Princeton, N. J.).

Pontrjagin, L. Classification of some skew products. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 322–325 (1945). [MF 14017]

Pontrjagin, L. Characteristic cycles. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 242–245 (1945). [MF 14023]

These papers present, in condensed form, results on the theory of skew products (fibre bundles). The first paper shows that skew products $P(S^{k-1}, B)$ of complexes B and spheres S^{k-1} may be classified, in terms of the mappings of B into the space $H(k, l)$ of oriented k-planes in Euclidean $(k+l)$-space, for any $l>\dim(B)$. This was proved independently by N. E. Steenrod [Ann. of Math. (2) **45**, 294–311 (1944); these Rev. **5**, 214].

The second paper classifies the above bundles in case $\dim(B)=4$. (The case $\dim(B)\leqq 3$ was taken care of by the reviewer [Bull. Amer. Math. Soc. **43**, 785–805 (1937)].) Taking the case $k=3$ for simplicity, the method is as follows. Two cocycles z^2 (mod 2) and z^4 (integral) are defined in H; $z^2 \cup z^2 \sim z^4$ (mod 4) [reviewer's notation]. Given any similar cocycles u^2, u^4 in B, there exists a corresponding mapping of B into H such that these are the antecedents of the z^i; hence also there is a corresponding skew product. If B has no even 4-dimensional cotorsion (the contrary case is too complicated to state here), these cocycles determine the skew product.

A second, more elementary, theorem states essentially the following. Let $P(A, B)$ be any skew product, direct over the n-dimensional part B^n of B. Choosing a representation f as a direct product over B^n determines an $(n+1)$-cocycle w_f^{n+1} in B, with coefficients in the group of spherical cycles of the group associated with P. Then the cohomology class of w_f^{n+1} is an invariant. Both theorems contradict a statement of the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 148–153 (1940), near the end of §4; these Rev. **1**, 220]. It is not easy to find who is wrong where. The simplest test case is furnished by the four-dimensional torus.
H. Whitney (Cambridge, Mass.).

Eckmann, Beno. Harmonische Funktionen und Randwertaufgaben in einem Komplex. Comment. Math. Helv. **17**, 240–255 (1945).

Let numerical values $f(x_i)$ be assigned on the vertices x_i of a subgraph R of a linear graph K. It is required to find f in $Q=K-R$ so that $f(x_i)$ for vertices x_i in Q equals the mean of its values on the vertices joined to x_i by an edge. It is shown that f always exists, and is unique if and only if R has at least one vertex in each component of K. This

is a generalization of a finite version of the Dirichlet problem. A similar theorem is given in n dimensions. The proof is based on the unique representation of a chain with real coefficients as a sum of a harmonic chain (which is both a cycle and a cocycle), a boundary and a coboundary.

H. Whitney (Cambridge, Mass.).

Rado, R. Two theorems on graphs. Ann. of Math. (2) 46, 429–467 (1945). [MF 13430]

The author defines an oriented graph Γ as a nonempty set G and a binary relation $a \rightarrow b$ between its elements. Real valued functions f defined on the elements of Γ are considered. Given $a \rightarrow b$ and given a real number δ an elementary transformation of f is defined by setting $f'(a) = f(a) - \delta$, $f'(b) = f(b) + \delta$ and $f'(x) = f(x)$ for $a \neq x \neq b$. The relation $f \rightarrow g$ means that it is possible to transform f into g by a finite or denumerable succession of elementary transformations. Given two functions f, h in Γ, a necessary and sufficient condition for the existence of a $g \geqq h$ such that $f \rightarrow g$ is that $\sum f(x) \geqq \sum h(x)$, $x \varepsilon A$, for any set A satisfying the following two conditions: (1) $x \rightarrow y$ and $y \varepsilon A$ imply $x \varepsilon A$; (2) $\cdots \rightarrow x_2 \rightarrow x_1 \rightarrow x_0 \varepsilon A$ implies that not all x_i are distinct. Reversing the inequalities and the arrows gives a necessary and sufficient condition for $f \rightarrow g$ for some $g \leqq h$. Combining the two conditions gives a necessary and sufficient condition for $f \rightarrow h$.

S. Eilenberg (Ann Arbor, Mich.).

MECHANICS

Tolotti, Carlo. Alcune proprietà degli assi d'equilibrio di Möbius. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 223–228 (1941) = Ist. Naz. Appl. Calcolo (2) no. 121. [MF 11947]

On étudie les axes de rotation de Moebius en applicant le symbolisme des homographies vectorielles. *B. Levi.*

Schwerdtfeger, H. On the representation of rigid rotations. J. Appl. Phys. 16, 571–576 (1945). [MF 13642]

This is an expository article. The fact that any group of two-rowed matrices with unit determinant can be represented by a special group of three-rowed matrices,

$$T = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \rightarrow D(T) = \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix},$$

is said to be not generally known and is the basis of one of the derivations of the spin representation. *A. Schwartz.*

This review is to be replaced by the one on p. 620.

Cârstoiu, I. Le mouvement à la Poinsot généralisé. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] 15, 61–69 (1944). [MF 13581]

The author studies the rotation of a rigid body about a fixed point O in the case in which $\mathbf{M}$, the vector moment of the applied forces about O, the angular velocity ω and the angular momentum $\mathbf{K}$ satisfy a relation of the form $\mathbf{M} = \lambda(\omega \times \mathbf{K})$. Here $\omega \times \mathbf{K}$ denotes the vector product and λ is a given scalar which may be a function of time. The properties of the motion are similar in many respects to those which hold in the familiar special case in which λ is zero. Among the novelties which appear in the general case are the following. The terminus of the vector $\mathbf{K}$ describes a spherical curve. The ellipsoid of inertia rolls without sliding on a variable plane which is constantly tangent to a fixed sphere. Some consideration is given to the case in which the body is not entirely rigid, but admits of internal movements which do not alter the principal moments of inertia.

L. A. MacColl (New York, N. Y.).

Weinstein, A., and Pounder, J. R. An electromagnetic analogy in mechanics. Amer. Math. Monthly 52, 434–438 (1945). [MF 13757]

The problem of the motion of a point charge influenced by static uniform electric and magnetic fields is shown to be identical with that of the motion of a heavy particle on a rotating earth, and a simple unified vector presentation is given.

P. Franklin (Cambridge, Mass.).

Kazakov, S. A. Calculation of trajectories of the movement of the centre of gravity of projectiles. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 129–138 (1945). (Russian. English summary) [MF 14038]

Nekrasov, A. I. Motion of heavy solids in a medium in accordance with the quadratic law of resistance. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 197–206 (1945). (Russian. English summary) [MF 14032]

The author submits a new mathematical solution for the well-known problem of the motion of a heavy solid on an incline, atmospheric pressure being constant. The main section treats of the motion of heavy bodies vertically from great heights. If the usual law of the distribution of density in accordance with the height of the atmosphere is accepted, the problem will be reduced to quadratures, which can be calculated only approximately. The author introduces a new transcendental function by means of which he derives the vector of velocity in a final form.

From the author's summary.

Bouzitat, Jean. Mécanique rationnelle. Démonstration directe du théorème de Painlevé. Revue Sci. (Rev. Rose Illus.) 80, 315–316 (1942). [MF 13812]

The result proved is that for certain types of mechanical systems there exists the Painlevé integral $T_2 - T_0 - U = \text{constant}$, made up of the terms of the second and zeroth degree in the kinetic energy T and the potential energy U.

P. Franklin (Cambridge, Mass.).

Schönberg, Mario. On a variational principle of dynamics. II. Anais Acad. Brasil. Ci. 17, 95–132 (1945). (Portuguese) [MF 13368]

Continuing the discussion of the variational principle introduced in the first part [same Anais 16, 293–310 (1944); these Rev. 6, 243] the author first studies its connection with integral invariants and differential forms, giving formulations in these terms equivalent to the principle. He then introduces an affine linear connection of a certain space associated with a dynamical system and finds that under that connection the curvature and the torsion of the space are zero. This corresponds to the fact that the corresponding group is Abelian and transitive. The connection between the infinitesimal transformations of this group and the integration of the Hamiltonian system is pointed out. The length of the paper is partly due to the fact that the cases when the Hamiltonian does not and does depend on time are treated separately.

G. Y. Rainich.

Hydrodynamics, Aerodynamics

Lichnerowicz, André. Sur l'invariant intégral de l'hydrodynamique relativiste. Ann. Sci. École Norm. Sup. (3) **58**, 285–304 (1941). [MF 13312]

As far as results are concerned, this paper follows in the main (with acknowledgment) the work of Eisenhart [Trans. Amer. Math. Soc. **26**, 205–220 (1924)] and Synge [Proc. London Math. Soc. (2) **43**, 376–416 (1937)]. The presentation is different, however, since the author uses the methods developed by Cartan in connection with integral invariants, and the basic assumptions are more general than those of Eisenhart and Synge, who assumed a pressure-density relation in the fluid. The author takes a weaker condition which states that, if the energy tensor is written $T_{\alpha\beta}=\rho u_\alpha u_\beta+\tau_{\alpha\beta}$, then there exists a function F such that

$$\nabla_\alpha(\tau_\beta{}^\alpha)=\rho\partial(\log F)/\partial x^\beta.$$

Among the results is the theorem of Bernoulli for steady motion [cf. the author's note in C. R. Acad. Sci. Paris **211**, 117–119 (1940); these Rev. **3**, 92]. The approach through integral invariants leads the author to define irrotational motion by the six conditions $\Omega_{\alpha\beta}=0$, where $\Omega_{\alpha\beta}$ is the skew-symmetric vorticity tensor. These conditions imply that the stream lines of an irrotational motion form a normal congruence of geodesics. *J. L. Synge* (Columbus, Ohio).

Lichnerowicz, André. Sur des théorèmes d'unicité relatifs aux équations gravitationnelles du cas intérieur. Bull. Sci. Math. (2) **65**, 54–72 (1941). [MF 13266]

This paper deals with irrotational motion of a perfect fluid in general relativity, an irrotational motion being one in which the stream lines form a normal congruence in space-time, and a perfect fluid being one for which the energy tensor is $T_{\alpha\beta}=\rho u_\alpha u_\beta-pg_{\alpha\beta}$, with a functional relationship between ρ and p. The condition of incompressibility is imposed; this means that the expansion of the congruence of stream lines is zero, and hence $\rho-p=$constant. The paper is chiefly concerned with uniqueness theorems, and culminates in a theorem which states that there exists at most one stationary irrotational motion of an incompressible fluid compatible with a given boundary and with Cauchy data on that boundary. [It appears to the reviewer that the proof is incomplete in the absence of an examination of the differential equations as to their normal character relative to the boundary.] *J. L. Synge.*

Sedov, L. I. On unsteady motions of a compressible fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 91–93 (1945). [MF 13746]

The author shows how to obtain some new solutions for the one-dimensional nonsteady flow of compressible fluids by use of the theory of dimensions. Two cases are considered: (I) the problem depends on only one dimensional constant a; (II) the problem depends on two dimensional constants a, b. Writing the constants in dimensional form as (1) $[a]=ML^kT^s$, $[b]=L^nT^s$, the expressions for the velocity, density, and pressure become (2) $v=(rV/t)$, $\rho=aR/(r^{k+3}t^s)$, $p=aP/(r^{k+1}t^{s+2})$, where V, R, P are independent of r and t in case (I), and possibly depend on the parameter $\lambda=br^{-n}t^{-s}$ in case (II). By substituting (2) into the two equations of motion and the equation of state (or an equivalent relation), there result, for case (I), three relations connecting V, P, R, k, s; for case (II), a differential equation of the form $d(P/R)/dV=(P/R)f(V, P/R)$. *N. Coburn* (Austin, Tex.).

Slezkin, N. A. Immersion of a disk in a viscous compressible medium. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 233–244 (1945). (Russian. English summary) [MF 14035]

Malavard, Lucien, et Siestrunck, Raymond. Sur l'étude, par expérimentation analogique, de grilles rectilignes de profils quelconques. C. R. Acad. Sci. Paris **220**, 810–812 (1945). [MF 14068]

Hughes, N. J. S. Stream expansion with discontinuity in velocity on the boundary. Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. 1978 (7768), 8 pp. (1944). [MF 13987]

The problem of the shape of the boundary of a two-dimensional stream which expands along an infinite duct, with boundaries on which the velocity is constant, apart from a single discontinuity, is considered mathematically. The shapes of the boundaries are calculated for velocity ratios of 1.5 and 2.0. Physical realization of this type of flow would require boundary layer suction at the points of discontinuity in velocity. *Author's summary.*

Panichkin, I. A. Concerning the influence of the boundary of a flow with circular cross section on the aerodynamic characteristics of a wing. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 171–178 (1945). (Russian. English summary) [MF 14041]

Landau, L. D., and Staniukovich, K. P. Determination of the flow velocity of the detonation products of condensed explosives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 271–274 (1945). [MF 14029]

Polubarinova-Kotschina, P. J. On the displacement of the oil-bearing contour. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 250–254 (1945). [MF 14025]

The author reconsiders the problem treated in her previous paper on the displacement of the boundary between two liquids in a porous medium [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 79–90 (1945); these Rev. **7**, 95] and discusses two cases in which the initial contour of the boundary is a cardioid or an infinite straight line. *I. Opatowski* (Chicago, Ill.).

Brun, Edmond, Fasso, Guy, et Vasseur, Marcel. Dépôt, sur un obstacle, de particules en suspension dans un courant fluide. C. R. Acad. Sci. Paris **218**, 341–344 (1944). [MF 13396]

A number of questions fundamental to the dynamics of particles suspended in a flowing fluid are discussed. It is assumed that the suspension does not influence the fluid flow, that gravity can be neglected, and that the fluid's velocity distribution can be derived from a potential. Under these assumptions the suspended particle is found to obey the differential equation $\rho' d\mathbf{U}'/dt-\varphi d\mathbf{U}/dt=(\rho u\mathbf{u}/d)\varphi(ud/\nu)$, where $\mathbf{U}'$ is the velocity vector of a fluid particle at a point M, $\mathbf{U}$ is the velocity vector of a suspended particle when it passes through point M, $\mathbf{u}=\mathbf{U}-\mathbf{U}'$, ρ and ρ' are the densities of the particle and of the fluid, ν is the kinematic viscosity of the fluid, d is the diameter of a particle and φ is the coefficient of resistance. The authors show that the path of a suspended particle, even in steady motion, may deviate substantially from the corresponding path of a fluid particle, and that a certain part of the suspended particles will come into contact with the solid obstacle and be "cap-

tured" by it. A coefficient is defined to characterize that percentage of particles, contained in a part of the undisturbed flow corresponding to the width of the obstacle, which are "captured" by the obstacle. The similarity laws of the phenomenon are discussed. *P. Neményi.*

Brun, Edmond, et Vasseur, Marcel. Dépôt, sur un obstacle, de particules en suspension dans un courant fluide. C. R. Acad. Sci. Paris **218**, 636–638 (1944). [MF 13473]

The argument of the paper reviewed above is followed up by a study of the percentage of "captured" particles as a function of the undisturbed velocity, and by the description of a graphical method for the determination of the path of a suspended particle and of the percentage relationship. *P. Neményi* (Pullman, Wash.).

Quraishy, Mohamed Saleh. The critical shear stress. J. Univ. Bombay (N.S.) **12**, part 3, 37–46 (1943). [MF 12942]

This study is a new attempt toward the unification of apparently divergent experimental results concerning the start of "general" (that is, of intense) movement in the movable bed of a canal. In addition to the well-known results of Gilbert, Casey and Kramer, little-known results of Shields, Srichamara and P. Y. Ho are considered. The author plots all these data in an (R, F) coordinate system, where V_* is the "terminal velocity" of the grains, d is the average grain diameter, ν is the kinematical viscosity, τ_{cr} is the critical shear stress and $R=V_*d/\nu$, $F=\tau_{cr}/(\rho_w V_*^2)$. He finds that all results for round (or roundish) grains can be well approximated in this system by a single curve; this same standard curve is found to be valid also for sand mixtures consisting partly of flat (slaty) grains, as long as R is moderate. Only at a considerable value of R does the grain shape start to cause substantial deviations from the standard curve. *P. Neményi* (Pullman, Wash.).

Elasticity, Plasticity

Tolotti, Carlo. Sulla più generale elasticità di 2° grado. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 20 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. **127**. [MF 11948]

Signorini a proposé comme forme possible du potentiel d'élasticité isotherme W une fonction linéaire et homogène des coefficients de Lamé λ, μ et d'une autre constante caractéristique c, avec coefficients dépendant simplement des invariants fondamentaux de la déformation; il a considéré en particulier le cas de $c=0$. L'auteur, en introduisant comme variables non homogènes les rapports $\kappa=\frac{1}{2}\lambda/(\lambda+\mu)$, $\nu=\frac{1}{2}c/(\lambda+\mu)$, détermine la région du plan (κ, ν) dans laquelle cette expression est positive et finie. *B. Levi.*

Tolotti, C. Deformazioni elastiche finite: onde ordinarie di discontinuità e caso tipico di solidi elastici isotropi. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **4**, 26 pp. (1943) = Ist. Naz. Appl. Calcolo (2) no. **153**. [MF 11943]

Dans ses Leçons sur la Propagation des Ondes [Hermann, Paris, 1903], Hadamard détermine la forme que doit avoir le potentiel d'élasticité afin qu'un axe de l'ellipsoïde de polarisation soit normal à la surface d'onde. L'auteur reprend la recherche utilisant le symbolisme du calcul vectoriel et la complète avec les conditions pour que l'ellipsoïde soit réel de rotation autour de cet axe. En examinant les conséquences statiques de ces hipothèses, il est conduit à les considérer conformes à la réalité. La forme du potentiel d'élasticité proposée par Signorini [cf. l'analyse précédente] ne satisfait pas toujours à ces conditions. *B. Levi.*

Tolotti, Carlo. Orientamenti principali di un corpo elastico rispetto alla sua sollecitazione totale. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1139–1162 (1942) = Ist. Naz. Appl. Calcolo (2) no. **145**. [MF 11945]

On peut se proposer de représenter la solution des équations de l'équilibre d'un solide élastique, dans le cas des déformations finies, en la développant en série de puissances d'un paramètre, de manière que le premier terme corresponde à supposer la déformation infiniment petite. Ce mode de représentation peut se heurter à une indétermination dans la calcul d'un quelconque des termes successifs; l'auteur appelle principale une orientation par rapport à laquelle cette exception ne se présente pas et donne une série de critères qui caractérisent ces orientations. Il applique sa théorie au cas du cylindre et de la sphère. *B. Levi* (Rosario).

Tolotti, Carlo. Le equazioni Lagrangiane della meccanica dei sistemi continui in coordinate generali. Atti Accad. Sci. Napoli (4) **13**, 1–9 (1942) = Ist. Naz. Appl. Calcolo (2) no. **150**. [MF 11944]

Représentation en coordonnées curvilignes générales des équations lagrangiennes de l'élasticité. *B. Levi.*

Charrueau, André. Sur des représentations planes du tenseur des contraintes dans un milieu continu. C. R. Acad. Sci. Paris **220**, 642–643 (1945). [MF 14060]

Green, A. E. A note on certain stress distributions in isotropic and aeolotropic materials. Proc. Cambridge Philos. Soc. **41**, 224–231 (1945). [MF 13421]

A solution is obtained by complex-variable methods for the distribution of generalized plane stress in an aeolotropic medium for the case of a concentrated force acting at some point of the interior of a half-plane the boundary of which is free of stress. *E. Reissner* (Cambridge, Mass.).

Sen, Bibhutibhusan. Stresses in an infinite plate due to isolated forces and couples acting near a circular hole. Philos. Mag. (7) **36**, 211–216 (1945). [MF 13345]

The paper is concerned with the distribution of stress in an infinite elastic slab with a circular hole, the load consisting of either a force which is parallel to the plane of the slab and acts at an interior point in the vicinity of the hole, or a couple which acts in the plane of the slab and is applied at such an interior point. *W. Prager.*

Sen, Bibhutibhusan. Problems of thin plates with circular holes. Bull. Calcutta Math. Soc. **37**, 37–42 (1945). [MF 13780]

The author studies the state of generalized plane stress which exists in an infinite plate when a single force (in the plane of the plate) or a couple (with axis normal to the plate) acts at a point which is near a circular hole of radius a. The method employed is to utilize the results found in Coker and Filon [A Treatise on Photo-Elasticity, Cambridge University Press, 1931, pp. 327, 362] for the generalized plane stresses in a complete plate under the action of this single force or couple, and to superpose an additional

force system which has suitable properties at the infinite boundary and which causes all normal stresses to vanish at the circular boundary. Two analytic functions $L(z)$ and $M(z)$ $(z=x+iy)$ are determined for this superposed stress system by employing some properties of the inverse of the loading point with respect to the given circular opening. The results include the problem of Bickley for a point load applied at the boundary of the hole in the plate. As a test of validity of the results one notes that the stresses are independent of Poisson's ratio for the loading couple but not for the single force. The hoop stresses always vanish on the boundary points which are on the diameter passing through the point where the couple is applied.

D. L. Holl (Ames, Iowa).

Fubini, Guido. Un problema sulle piastre su cui agisce un carico concentrato e sue generalizacioni analitiche. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 13–18 (1945). [MF 13720]

A method is outlined for calculating the displacement of a thin plate having a general boundary B and loaded uniformly on a circular area internal to B. *I. Opatowski.*

Courbon, M. Sur la solution donnée par Navier au problème de la flexion des plaques rectangulaires. Génie Civil **120**, 41–42 (1943). [MF 13798]

Discussion of a note by E. Doucet [Génie Civil **119**, 316–317 (1942); cf. these Rev. **6**, 139].

Smith, R. C. T. The buckling of flat plywood plates in compression. Austral. Counc. Aeronaut. Rep. **ACA-12**, 28 pp. (1944). [MF 13415]

A short discussion is given of the elastic behavior of wood and plywood considered as orthotropic materials [see also H. W. March, Physics **7**, 32–41 (1936); U. S. Forest Products Laboratory Reports nos. 1312, 1316, 1316B, 1316C, 1316F]. The problem of finding the buckling load of a rectangular orthotropic plate under uniform compression in one direction only is solved for each of the following three boundary conditions: (1) the two loaded edges clamped and the remaining edges simply supported, (2) the loaded edges simply supported and the remaining edges clamped, (3) all edges clamped. It is assumed that the edges are parallel to the axes of elastic symmetry in the plane of the plate. Problem (1) is solved by standard methods of procedure and a family of curves is presented for determining the buckling load of a plate of given dimensions and elastic properties. The results of a series of confirmatory tests are given. For problem (2) a method used by D. M. A. Leggett [Proc. Roy. Soc. London. Ser. A. **162**, 62–83 (1937)] is employed; it leads finally to an equation which expresses the vanishing of an infinite series involving the buckling load. This equation is solved by successive approximation. As in problem (1), the results are presented in a family of curves. The Rayleigh-Ritz method is used to obtain an approximate solution of problem (3). The functions used are due to S. Iguchi [Eine Lösung für die Berechnung der Biegsamen Rechteckigen Platten, Springer, Berlin, 1933] and can be obtained by Leggett's method. The results are again expressed in a family of curves. *H. W. March.*

Rabotnov, J. N. Fundamental equations of the theory of shells. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 87–90 (1945). [MF 13745]

The fundamental equations for an elastic thin shell are derived and expressed in intrinsic coordinates u, v on the undisplaced median surface of the shell. *H. S. Tsien.*

Rabotnov, J. N. On the boundary zone equations in the theory of shells. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 329–331 (1945). [MF 14019]

The general fundamental equations given in the paper reviewed above are reduced to simple forms under the assumption that the region considered is of the order of the geometrical mean of the thickness and the radius of curvature of the shell. A physical interpretation of the resultant equations is given. These equations are shown to reduce to familiar forms for symmetrical and symmetrically loaded rotation shells. *H. S. Tsien* (Pasadena, Calif.).

Fadle, J. Eine einfache Ableitung der Gleichgewichtsbedingungen des Schalenproblems. Ing.-Arch. **14**, 413–422 (1944). [MF 13377]

The author derives the fundamental equations for thin shells of arbitrary shape with varying thickness. The equilibrium equations for forces and moments are first written in terms of the Gaussian fundamental quantities of second order for the deformed median surface of the shell. These fundamental quantities of second order for the deformed surface are then expressed in turn as functions of the Gaussian fundamental quantities for the undeformed median surface of the shell and the displacement vectors. Here the second order terms involving products of the displacements and their partial derivatives in the fundamental quantities for the deformed surface are neglected. Simpler expressions are obtained for the case where the extension of the median surface itself during deformation is negligible. The problem was treated previously by F. Reutter [Z. Angew. Math. Mech. **22**, 87–98 (1942); these Rev. **4**, 230]. *H. S. Tsien* (Pasadena, Calif.).

Vlasov, V. Z. Calculations of the thin-walled prismatical shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 361–394 (1944). (Russian. English summary) [MF 12389]

Continuing his previous studies [see the same journal **8**, 109–140 (1944); these Rev. **7**, 42], the author investigates thin-walled prismatic shells, replacing the two-dimensional continuous medium by a frame of beams. He assumes that the displacements $u(z, s)$ and $v(z, s)$ of the beams in the longitudinal and in the transversal directions (z and s directions, respectively) can be written in the form

$$u(z, s)=\sum_{i=1}^{m} U_i(z)\phi_i(s), \quad v(z, s)=\sum_{k=1}^{n} V_k(z)\psi_k(s),$$

where U_i and V_k are unknown functions and ϕ_i and ψ_k have to be chosen suitably. Applying classical methods of the theory of beams, he obtains for U_i and V_k a system of ordinary differential equations with constant coefficients, which can then be solved. Shells with rectangular, triangular, U-shaped and fairly complicated cross-sections are considered. The method is applied to the study of the vibrations of multiply connected shells. The results obtained are compared with exact solutions. *S. Bergman.*

Bromberg, E., and Stoker, J. J. Non-linear theory of curved elastic sheets. Quart. Appl. Math. **3**, 246–265 (1945). [MF 13536]

The theory is based upon the assumptions (1) the strains due to the normal stress on elements parallel to the surface of the shell are small enough to be neglected safely; (2) all stresses are constant over the thickness of the shell and are linear functions of the strain. Three types of treatment of the problem result from further specialization. Case (a) results from the direct linearization of the differential equa-

tion. For a flat sheet, this case is trivial as it requires the normal pressure load p to be zero. For curved sheets, this is the usual membrane theory of shells, where the restrictions on the displacements of the sheet at the boundary generally cannot be satisfied. Case (b) results from the assumption of large tension in the sheet but small normal pressure p. Here only the displacement normal to the sheet can be specified at the boundary. Case (c) is the nonlinear theory of sheets where all the displacements at the boundary can be specified. For flat sheets, this is the Föppl-Hencky theory. The authors are mainly concerned with the proper formulation of the corresponding theory for curved rotationally symmetric sheets. The particular case of flat spherical sheets is treated in some detail with a discussion of the "boundary layer solution" for the edges. Graphs are presented comparing the numerical solution, the boundary layer solution and the usual membrane solution. Satisfactory agreement between the first two solutions is shown. *H. S. Tsien* (Pasadena, Calif.).

Platrier, Charles. Le problème de Barré de Saint-Venant dans certains milieux anisotropes. Ann. Ponts Chaussées **1942** (112e année), 277–306 (1942). [MF 13714]

Platrier, Charles. Complément et correction d'une méthode de résolution des problèmes d'équilibres élastiques. Ann. Ponts Chaussées **1941** (111e année), 449–454 (1941). [MF 13778]

The author extends his previous general discussion [same Ann. **1940 II** (110e année), 251–292 (1940); these Rev. **7**, 40] of a method of solution of problems associated with the name of Saint-Venant to certain anisotropic materials and to such materials subjected to a certain type of initial stress. He corrects an error in the paper cited above. The error arose from overlooking the fact that certain functions employed were multiple-valued. *H. W. March.*

Goelzer, A. Équilibre d'un solide élastique prismatique soumis à des efforts extérieurs tangentiels. Ann. Ponts Chaussées **1943** (113e année), 155–174 (1943). [MF 13713]

Seth, B. R. Finite strain in aelotropic elastic bodies. I. Bull. Calcutta Math. Soc. **37**, 62–68 (1945). [MF 13783]

It is assumed that the stress-strain relations are linear and in the expressions for the components of strain the quadratic terms are retained. The following problems are solved: (1) cylinder under uniform tension, (2) uniform hydrostatic pressure, (3) thick rectangular plate bent by uniformly distributed terminal couples. *E. Reissner.*

Bisshopp, K. E., and Drucker, D. C. Large deflection of cantilever beams. Quart. Appl. Math. **3**, 272–275 (1945). [MF 13539]

It is assumed that the curvature is proportional to the bending moment and that the length of the beam does not change during bending. The differential equation of the elastic line is set up by expressing its curvature in terms of the length of arc and slope. The vertical deflection of the end of the cantilever is found in terms of complete and incomplete elliptic integrals of the first and second kinds. Curves are presented for the vertical and horizontal displacements of the end of the cantilever. This problem has also been treated by H. J. Barten [cf. the following review]. *H. W. March* (Madison, Wis.).

Barten, H. J. Corrections to my paper on the deflection of a cantilever beam. Quart. Appl. Math. **3**, 275–276 (1945). [MF 13540]

The author corrects an error in a previous paper [same Quart. **2**, 168–171 (1944); these Rev. **6**, 83] and gives corrected curves representing the results. *H. W. March* (Madison, Wis.).

Mandel, J. Sur diverses instabilités d'une tige: torsion, flexion, torsion avec effort longitudinal. Ann. Ponts Chaussées **1944** (114e année), 461–484 (1944). [MF 13712]

Rosenblatt, Alfred. On the problem of the elastic arch subjected to constant pressures on the extrados and on the intrados. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 245–254 (1945). (Spanish) [MF 13731]

Strub, R. General approximative formulae for the lateral vibrations of a bar with built-in ends. Sulzer Tech. Rev. **1945**, no. 1, 127–129 (1945). [MF 14078]

Asymptotic formulas are derived for the natural frequencies and the positions of the nodes of a prismatic beam with built-in ends. The errors introduced by the use of these approximate formulas do not exceed 0.8% and 0.4%, respectively, for any mode of vibration. *W. Prager.*

Foix, A. Ondes sphériques transversales solides polarisées rectilignement. J. Phys. Radium (8) **1**, 311–316 (1940). [MF 13983]

Purely distortional spherical waves in an isotropic elastic continuum are considered (each spherical surface as a whole oscillating parallel to the Z-axis). They satisfy the second order vector wave equation and, if divergent, are formally identical with the corresponding electromagnetic waves. They are different from the so-called natural spherical waves. Reference is made to the Huygens-Fresnel principle, and the treatment of diffraction is indicated. *H. G. Baerwald* (Cleveland Heights, Ohio).

Ekstein, H. High frequency vibrations of thin crystal plates. Phys. Rev. (2) **68**, 11–23 (1945). [MF 12875]

This is an investigation of the modes of vibration of plates of finite extent whose elastic constants correspond to those of quartz crystals. Satisfaction of the boundary conditions is obtained for the larger surfaces, approximated on one pair of edges, but depends on the smallness of certain elastic constants on the remaining edges. The frequencies predicted by the theory are shown to be in good agreement with experimental evidence. *G. F. Carrier.*

Parodi, Maurice. Propagation d'ondes élastiques planes dans un milieu hétérogène. C. R. Acad. Sci. Paris **218**, 69–71 (1944). [MF 13451]

Since the equation for the wave propagation of an elastic wave is the same as that of an electromagnetic wave, the usual electrical procedure can be used in the elastic case. *N. Levinson* (Cambridge, Mass.).

Petrowsky, I. G. On the propagation velocity of discontinuities of the displacement derivatives on the surface of an unhomogeneous elastic body of arbitrary form. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 255–258 (1945). [MF 14026]

Dix, C. H., Fu, C. Y., and McLemore, Ethel W. Rayleigh waves and free surface reflections. Quart. Appl. Math. **3**, 151–156 (1945). [MF 12651]

A plane compressional incident wave is reflected by the plane boundary of a semi-infinite, isotropic, homogeneous, perfectly elastic solid. The reflection coefficient has a well-known expression as a ratio of two expressions involving incident and reflective amplitudes and velocities of compressional and shear waves and angles. The vanishing of the denominator yields the velocity of the Rayleigh wave. The authors make a graphical and numerical study of this formula. *D. G. Bourgin* (Urbana, Ill.).

Carrier, G. F. On the vibrations of the rotating ring. Quart. Appl. Math. **3**, 235–245 (1945). [MF 13535]

Thin rings rotating with constant speed about their geometric axes are analyzed for characteristic frequencies at which bending vibrations occur. The ring may be free or it may be supported at n evenly spaced intervals such that there is no relative tangential displacement of the points of support. The rotating points of support may admit either radial or no radial movement. In the supported ring the physical problem is that of determining what periodic forces applied at the supports are capable of sustaining a motion wherein the support points have no relative tangential displacement at any time. Solutions are found satisfying this condition for particular values of the frequencies or an associated eigenvalue. The author asserts that the set of solutions has the property of completeness and that the motion may be fully described for any initial conditions which have the appropriate periodicity in the angular variable. The treatment also includes forced vibrations and elastically supported rings. *D. L. Holl* (Ames, Iowa).

Rakhmatulin, K. A. Propagation of a wave of unloading. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 91–100 (1945). (Russian. English summary) [MF 13507]

The paper is concerned with the propagation of deformation along a semi-infinite rod subjected to a sudden pressure on its near end. The applied pressure decreases monotonically with time and approaches some constant value. The problem for the case when the pressure does not exceed the elastic limit of the material was solved by Fourier. This paper treats plastic deformations in which the laws of loading and unloading are different. *I. S. Sokolnikoff.*

Alfrey, Turner, and Doty, Paul. The methods of specifying the properties of viscoelastic materials. J. Appl. Phys. **16**, 700–713 (1945). [MF 14090]

Seven models (Voight model, Maxwell model, operator equation, mechanical impedance function, creep curve, relaxation curve, and dynamic modulus function) of specifying viscoelastic behavior are discussed. A number of exact relations between these methods of specification are worked out in detail. *From the authors' summary.*

Išlinskiĭ, A. Yu. Equations of deformation of the incompletely elastic and viscoplastic states. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 34–45 (1945). (Russian) [MF 13704]

The author discusses numerous scalar stress-strain relations representing the mechanical behavior of elastic-plastic, visco-elastic and visco-plastic solids under pure shear. The mechanical behavior of these solids is illustrated by models consisting of springs and dashpots with solid and viscous friction. *W. Prager* (Providence, R. I.).

L'Hermite, Robert. La déformation plastique visqueuse des solides. C. R. Acad. Sci. Paris **220**, 439–440 (1945). [MF 13958]

The author discusses the mechanical behavior of visco-elastic solids in simple tension and suggests standard tests for the determination of the constants which appear in the stress-strain relation. *W. Prager* (Providence, R. I.).

Sokolovsky, W. W. Plastic equilibrium equations of a plane stressed state. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 111–128 (1945). (Russian. English summary) [MF 13509]

The paper is concerned with states of plane stress in a plastic material which deforms in accordance with Hencky's theory [Z. Angew. Math. Mech. **4**, 323–334 (1924)]. The mathematical treatment follows closely that given by S. Christianovitch [Rec. Math. [Mat. Sbornik] N.S. **1(43)**, 511–534 (1936)]. *W. Prager* (Providence, R. I.).

Ilyushin, A. A. Finite relationship between stresses and its relation with deformations in theory of shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 101–110 (1945). (Russian. English summary) [MF 13508]

As the Russian title (A finite relation between the forces and moments and their connection with the deformations in the theory of shells) indicates more clearly than the title of the short English summary, the paper is concerned with the implications which the local yield condition for the stress components at a point of a plastic shell has for its stress resultants. The initial steps of the analysis are carried through for an incompressible material with strain hardening, but the analysis is completed only in the case of a perfectly plastic material which follows the v. Mises yield condition. The following examples are discussed: (1) plate stressed in its plane, (2) bent plate and (3) cylindrical shell under axially symmetric loading. In the case of a bent plate, for instance, the bending moments M_{11}, M_{22} and the twisting moment M_{12} are found to be connected by the relation $M_{11}^2+M_{22}^2-M_{11}M_{22}+3M_{12}^2=\text{constant}$.

W. Prager (Providence, R. I.).

Ilyushin, A. A. Relation between the theory of St. Venant-Levy-Mises and the theory of small elastico-plastic deformations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 207–218 (1945). (Russian. English summary) [MF 14285]

The author investigates the conditions under which the theory of plastic flow (St. Venant-Levy-Mises) and the theory of plastic deformation (Hencky) furnish identical predictions concerning the mechanical behavior of plastic solids. According to the first theory the instantaneous stress is determined by the instantaneous velocity strain, while the second lets the instantaneous stress depend on the instantaneous strain only. [The stress-strain relations discussed take account of strain hardening and are therefore somewhat more general than the relations considered by the authors mentioned.] A "tensor of similitude" is introduced; its components are obtained by dividing the components s_{ij} of the stress deviation by the invariant $(3s_{ij}s_{ij}/2)^{\frac{1}{2}}$. It is shown that the predictions of the two theories of plasticity will coincide if, for each element of the plastic body, the tensor of similitude remains constant during the deformation. *W. Prager* (Providence, R. I.).